Pure Mathematics 30

Archived Bulletin Information

Curriculum Standards and Example Questions

Diploma Examinations Program

Alberta EDUCATION

This document was written primarily for:

Students	✓
Teachers	✓ of Pure Mathematics 30
Administrators	✓
Parents	
General Audience	
Others	

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Introduction

Many of the examples shown in this document were chosen to illustrate the intent of particular outcomes of Pure Mathematics 30 but will not necessarily be assessed on a diploma examination in the manner shown. Some examples were developed and validated by classroom teachers of mathematics but have not been validated with students. Other examples were taken from the previous Pure Mathematics 30 diploma examinations.

To meet the outcomes of Pure Mathematics 30, students will need access to an approved graphing calculator. In most classrooms, students will use a graphing calculator daily. Refer to the calculator policy in the *Pure Mathematics 30 Information Bulletin* or go to the Alberta Education web site, www.education.gov.ab.ca, for a list of approved graphing calculators.

This document presents a final revision of the curriculum standards. If you have comments or questions regarding this document, please contact Ken Marcellus by e-mail at Ken.Marcellus@gov.ab.ca, by phone at (780) 427-0010, or by fax at (780) 422-4454.

Learner Assessment would like to recognize and thank the many teachers throughout the province who helped to prepare this document. We would also like to thank the Curriculum Branch and the Learning Technologies Branch for their input and assistance in reviewing these standards.

Contents

Transformations of Functions

General Outcome

Perform, analyze, and create transformations of functions and relations that are described by equations or graphs.

General Notes:
- Analysis of domain, range, intercepts, and asymptotes will be addressed in these outcomes.
- The use of the notation $y - k = f(x)$ is intended to emphasize that this is a transformation on y; this notation is particularly useful when transforming conic sections. This notation can be expressed as $y = f(x) + k$ for use on the graphing calculator.
- Students should be able to perform, analyze, and describe the transformation of any relation given its graph.
- To develop algebraic transformations, one should begin with functions that have been previously studied; i.e., $y = x$, $y = x^2$, $y = x^3$, $y = |x|$, $y = \frac{1}{x}$, $y = \sqrt{x}$.
- By the end of the Pure Mathematics 30 course, students should be able to perform, analyze, and describe transformations on the additional functions and relations studied in the course; i.e., trigonometric functions, logarithmic functions, quadratic relations (conic sections), and exponential functions.
- Students should recognize that invariant points are defined as points that do not move when a function undergoes transformation(s).
- Knowledge of $y = |f(x)|$ is no longer part of the curriculum; however, students will be expected to transform $y = |x|$.

Specific Outcomes

Specific Outcome 1.1

Describe how various translations of functions affect graphs and their related equations:
- $y = f(x - h)$
- $y - k = f(x)$

[C, T, V]

1.1 Note:
- For $y = f(x)$, translations occur when x is replaced with $(x - h)$ and y is replaced with $(y - k)$.

(See examples 1 and 2)

Specific Outcome 1.2

Describe how various stretches of functions (compressions and expansions) affect graphs and their related equations:

- $y = af(x)$
- $y = f(kx)$

[C, T, V]

1.2 Notes:

- Given that vertical stretches by a factor of $|a|$ occur when y is replaced with $\frac{1}{a}y$, the transformed equation can be written as $\frac{1}{a}y = f(x)$.
- Given that horizontal stretches by a factor of $\left|\frac{1}{k}\right|$ occur when x is replaced with kx, the transformed equation can be written as $y = f(kx)$.
- Students must be able to indicate the line about which a given function is stretched vertically or horizontally.
- A stretch about a horizontal or vertical line other than the x- or y-axis should be limited to sketching a stretch of a given graph and interpreting the characteristics, properties, and equation of the stretched graph.
- Students may find it useful to describe these transformations in this manner: "The graph of the function is stretched vertically/horizontally about the line __________ by a stretch factor of __________."

(See examples 3 and 4)
(Also see example 10c in Conics)

Specific Outcome 1.3

Describe how reflections of functions in both axes and in the line $y = x$ affect graphs and their related equations:

- $y = f(-x)$
- $y = -f(x)$
- $y = f^{-1}(x)$

[C, T, V]

1.3 Notes:

- The notation $y = f^{-1}(x)$ is equivalent to $x = f(y)$, the inverse of $y = f(x)$, not to the reciprocal of a function. Also, $y = f^{-1}(x)$ should only be used if the inverse is a function.
- Some questions should combine horizontal and vertical reflections.

(See examples 5 and 6)

Specific Outcome 1.4

Using the graph and/or the equation of $f(x)$, describe and sketch $\frac{1}{f(x)}$. [C, T, V]

1.4 Note:

- Domain, range, asymptotes and invariant points should be discussed. This outcome is a continuation of rational functions studied in Pure Mathematics 20.

(See examples 7, 8, and 9)

Specific Outcome 1.5

Describe and perform single transformations and combinations of transformations on functions and relations. [C, T, V]

1.5 Notes:

- On the diploma examination, transformations of relations may be assessed as part of the conics unit.
- Students must realize that a transformed function rewritten as $y = a \bullet f[b(x-h)] + k$ or as $\frac{1}{a}(y-k) = f[b(x-h)]$ will be drawn or described by using stretches/reflections, followed by translations.

(See example 10)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• determine and describe the effects of a transformation on the domain, the range, and the intercepts, and identify invariant points of the relation or function	
• recognize the graph associated with an algebraic function (as listed in the general notes)	• determine the equation of a transformed function given its graph
• use appropriate notation to describe a transformation	
• perform, analyze, and describe a horizontal translation $(y = f(x - h))$ **and/or** a vertical translation $(y - k = f(x))$ graphically or algebraically, given the function f in equation or graphical form	
• perform, analyze, and describe a horizontal stretch about the *y*-axis $(y = f(bx),\ \text{where}\ b > 0)$ or a vertical stretch about the *x*-axis $(y = af(x),\ \text{where}\ a > 0)$ graphically or algebraically, given the function $y = f(x)$ in equation or graphical form	• perform, analyze, and describe a horizontal stretch and reflection about the *y*-axis $(y = f(bx),\ \text{where}\ b < 0,\ b \neq -1)$ **or** a vertical stretch and reflection about the *x*-axis $(y = af(x),\ \text{where}\ a < 0,\ a \neq -1)$, graphically or algebraically, given the function $y = f(x)$ in equation or graphical form
• perform, analyze, and describe a reflection in the *x*-axis $(y = -f(x))$ and/or in the *y*-axis $(y = f(-x))$, or in the line $y = x$ $(y = f^{-1}(x))$ graphically or algebraically, given the function f in equation or graphical form	• perform, analyze, and describe a horizontal or vertical stretch about a line other than the *x*- or *y*-axis, given the function $y = f(x)$ in graphical form
• sketch and describe $y = \dfrac{1}{f(x)}$ given the equation or graph of $f(x)$	
• analyze domain, range, vertical asymptotes, and invariant points of $y = \dfrac{1}{f(x)}$	• analyze and determine the equation of $y = \dfrac{1}{f(x)}$, given the graph of $f(x)$
• perform, analyze, and describe combinations of transformations on functions or relations not involving reflections	• perform, analyze, and describe combinations of transformations on functions or relations involving reflections
• participate in and contribute toward the problem-solving process for problems involving the transformations of functions and relations studied in Pure Mathematics 30	• complete the solution to problems involving the transformations of functions and relations studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *standard of excellence*.

Numerical Response

1. As a result of the transformation of the graph of $y = x^3$ into the graph of $y - 4 = (x - 3)^3$, point (3, 27) becomes point (6, y). The value of y is __________.

Solution: 31

$$\begin{aligned} y - 4 &= (6 - 3)^3 \\ y - 4 &= 27 \\ y &= 31 \end{aligned}$$

or

$y = (x - 3)^3 + 4$ is $y = x^3$ with a horizontal shift of 3 units right and a vertical shift of 4 units up. Therefore, 27 becomes 31.

2. The graph of $y = f(x) = b^x$, where $b > 1$, is translated such that the equation of the new graph is expressed as $y - 2 = f(x - 1)$. The range of the new function is

* **A.** $y > 2$
B. $y > 3$
C. $y > -1$
D. $y > -2$

Solution:

The range of $y = b^x$ is $y > 0$. The translated graph has been shifted up 2 units and so the new range is $y > 2$.

3. If y is replaced by $\frac{1}{2}y$ in the equation $y = f(x)$, then the graph of $y = f(x)$ will be stretched

A. horizontally about the y-axis by a factor of $\frac{1}{2}$

B. horizontally about the y-axis by a factor of 2

C. vertically about the x-axis by a factor of $\frac{1}{2}$

* **D.** vertically about the x-axis by a factor of 2

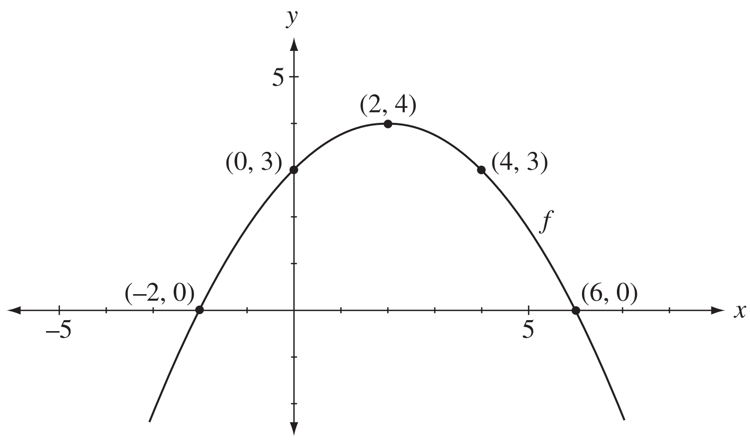
Solution:

$$\begin{aligned} \frac{1}{2}y &= f(x) \\ y &= 2 \bullet f(x) \end{aligned}$$

Therefore, the graph of $y = f(x)$ will be vertically stretched about the x-axis by a factor of 2.

Use the following information to answer the next question.

The partial graph of $y = f(x)$ is shown below.



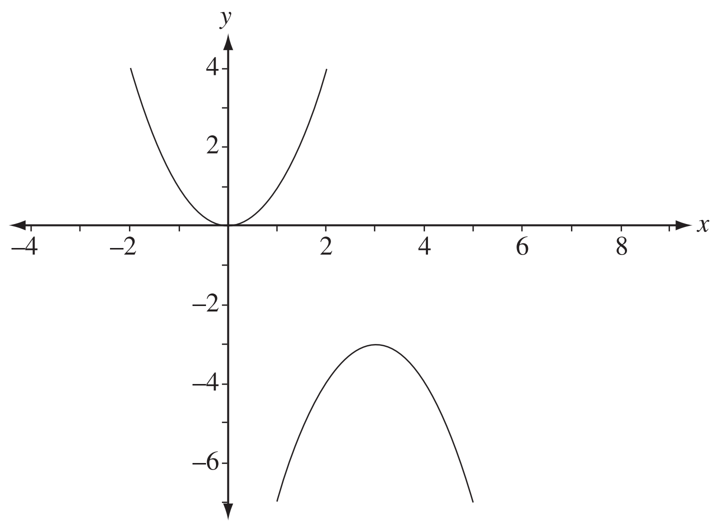
SE **4.** If the above graph is stretched vertically about the line $y = 3$ by a factor of $\frac{1}{2}$, then it will have

* **A.** invariant points at (0, 3) and (4, 3)
B. invariant points at (–2, 0) and (6, 0)
C. an invariant point at (1, 4)
D. an invariant point at (0, 3)

Solution:

Points on the line $y = 3$ are invariant.

SE **5.** A transformed image of a partial graph of $y = x^2$ is shown below. The two partial graphs are congruent and the vertex of the transformed image is at $(3, -3)$. Determine the equation of the transformed image. Explain your answer.



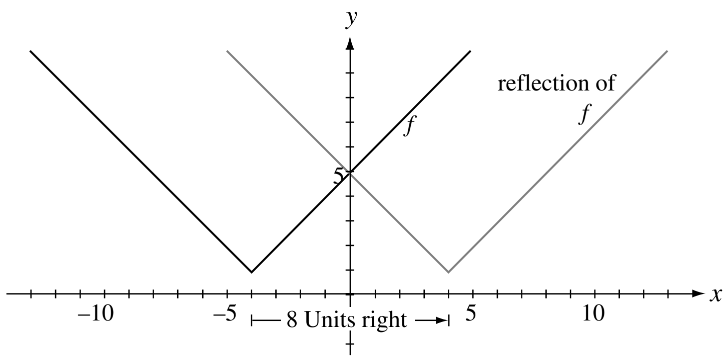
Solution:

Since the graph opens downward, it is a reflection of $y = x^2$ in the x-axis. The graph has been translated horizontally 3 units in the positive x direction and vertically 3 units in the negative y direction; therefore, the equation is $y + 3 = -1(x - 3)^2$ or $y = -1(x - 3)^2 - 3$. Because the graphs are congruent, the absolute value of a remains at 1.

SE **Numerical Response**

6. The graph of $y = f(x)$, where $f(x) = \left|x + 4\right| + 1$, is reflected in the y-axis. This produces the same results as would translating the graph of $y = f(x)$ to the right by __________ units.

Solution: 8



The graph of $f(x) = \left|x + 4\right| + 1$ is 4 units to the left of the y-axis; therefore, the reflected graph is 4 units to the right of the y-axis. The reflected graph is 8 units to the right of the original graph.

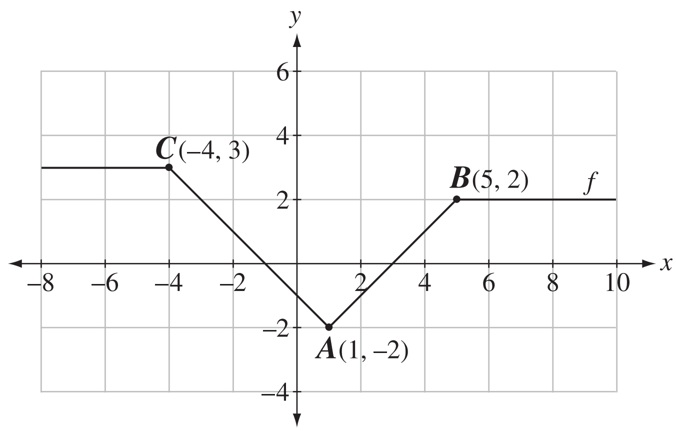
or

replace x with $-x$

$$
\begin{aligned}
y &= \left|-x + 4\right| + 1 \\
&= \left|-(x - 4)\right| + 1
\end{aligned}
$$

Use the following information to answer the next question.

The partial graph of $y = f(x)$ with all of its x-intercepts is shown below.



The x-intercepts of f are at $(-1, 0)$ and $(3, 0)$. The y-intercept of f is at $(0, -1)$.

7. **a.** The graph of $y = f(x)$ is transformed by stretching it horizontally about the y-axis by a factor of $\frac{1}{2}$, then translating it horizontally left by 1 unit, and finally translating it vertically up by 3 units. Express the resulting transformed function in terms of $f(x)$.

SE **b.** The graph of $y = f(x)$ is reflected about the y-axis and then reflected about the x-axis. State the new x- and y-intercepts of the transformed graph.

c. State the equation of the vertical asymptotes and the coordinates of the invariant points when $y = f(x)$ is transformed to $y = \dfrac{1}{f(x)}$.

Solutions:

a. The horizontal stretch by a factor of $\frac{1}{2}$ $\quad y = f(2x)$

The translation left 1 $\quad y = f[2(x+1)]$

The translation up 3 units $\quad y - 3 = f[2(x+1)]$

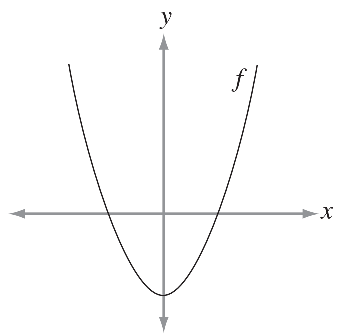
or $y = f[2(x+1)] + 3$

b. The y-intercept is at $(0, 1)$ and the x-intercepts are at $(1, 0)$ and $(-3, 0)$.

c. The vertical asymptotes are $x = -1$ and $x = 3$.
The invariant points are at $(0, -1)$, and approximately $(-2, 1)$, $(2, -1)$, and $(4, 1)$.

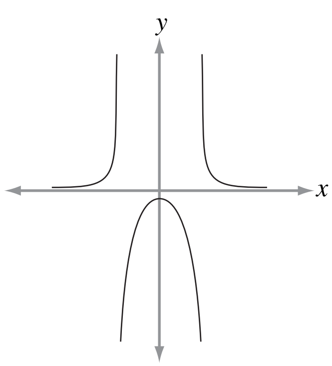
Use the following information to answer the next question.

The partial graph of $y = f(x)$ is shown below.

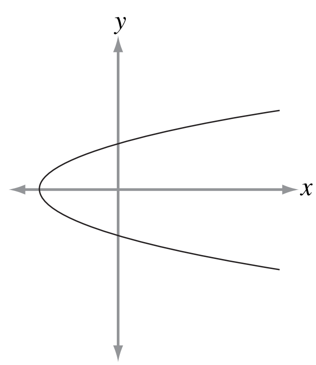


8. Which of the following graphs represents $y = \dfrac{1}{f(x)}$?

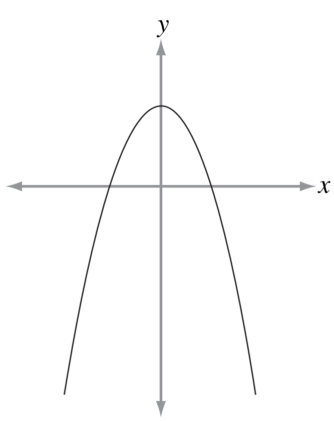
* **A.**



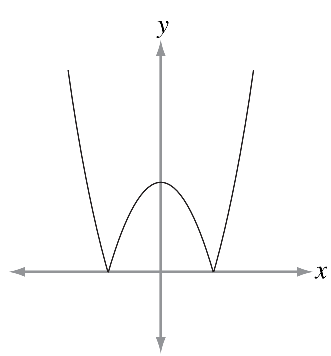
B.



C.



D.

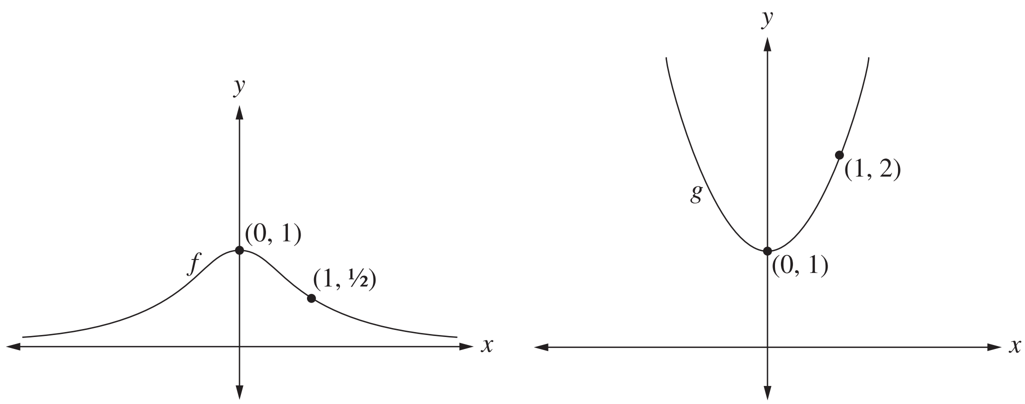


Solution:

A is the only alternative that has two vertical asymptotes, a negative *y*-intercept, and no *x*-intercepts.

Use the following information to answer the next question.

The partial graphs of two functions, f and g, are shown below.



9. If graph g is a transformation of graph f, then the equation that generates graph g is

A. $g(x) = f(-x)$

B. $g(x) = f(x)$

C. $g(x) = -f(x)$

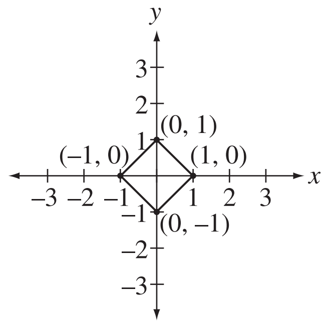
* **D.** $g(x) = \dfrac{1}{f(x)}$

Solution:

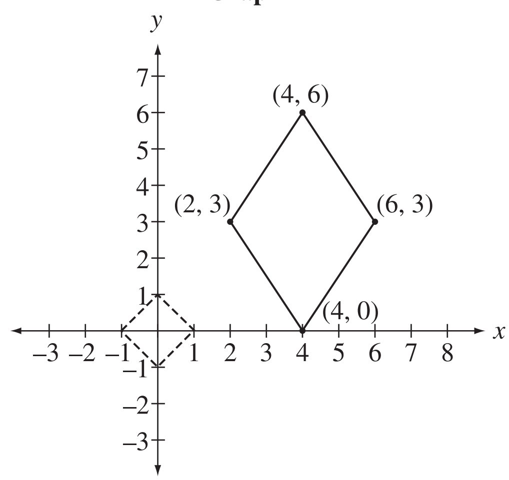
Since the point (0, 1) stays fixed (invariant) and the point $\left(1, \frac{1}{2}\right)$ becomes point (1, 2), the y-coordinates are reciprocals of each other. Therefore, the functions are also reciprocals of each other.

Use the following graphs to answer the next question.

Graph 1



Graph 2



10. The graph of a diamond has a centre at point (0, 0) and goes through the points (–1, 0), (0, 1), (1, 0), and (0, –1), as shown in graph 1 above. Describe the series of transformations required to transform graph 1 to graph 2 above, which is a diamond with centre at point (4, 3), a vertical length of 6 units measured parallel to the y-axis, and a horizontal width of 4 units measured parallel to the x-axis.

Solution:

Graph 1 has to be stretched vertically about the x-axis by a factor of 3 and stretched horizontally about the y-axis by a factor of 2. The graph is then translated 4 units to the right and 3 units up.

or

Graph 1 is translated 4 units to the right and 3 units up. The graph is then stretched vertically about the line $y = 3$ by a factor of 3 and stretched horizontally about the line $x = 4$ by a factor of 2.

Exponents, Logarithms, and Geometric Series

General Outcome

Generate and analyze exponential patterns.

Solve exponential and logarithmic equations and verify identities.

Represent and analyze exponential and logarithmic functions, using technology as appropriate.

General Note:

- The window format for graphing calculators is x: $[x_{min}, x_{max}, x_{scl}]$ y: $[y_{min}, y_{max}, y_{scl}]$.

Specific Outcomes

Specific Outcome 2.1

Derive and apply expressions to represent general terms and sums for geometric growth and to solve problems. [CN, R, T]

2.1 Notes:

- Revisit geometric sequences from Pure Mathematics 10. The general term was not an outcome in Pure Mathematics 10 and should be derived in class. Students are expected to apply the general term to examples and problems.
- The use of sigma notation to represent geometric series is part of this outcome.
- This outcome is not intended to include the recursive definition of a geometric sequence.

(See examples 1, 2, 3, 4, and 5)

Specific Outcome 2.2

Connect geometric sequences to exponential functions over the natural numbers. [E, R, V]

2.2 Notes:

- The use of discrete data and discrete graphs is the focus of this outcome. The solutions for n are over the natural numbers. The use of tables and elements from the graph can be incorporated with technology to solve these problems.
- The exponent is to be a natural number.
- Teachers may wish to graph discrete data or to have students collect real-world data and use exponential regression to find an equation that models their data.
- To solve compound interest problems, $A = P(1 + i)^n$ could be used as an example of a geometric sequence.

(See examples 6 and 7)

Specific Outcome 2.3

Solve exponential equations having bases that are powers of one another. [E, R]

(See example 8)

Specific Outcome 2.4

Use the laws of exponents and logarithms to

- solve and verify exponential equations and identities
- solve logarithmic equations
- simplify logarithmic expressions

[R]

2.4 Notes:

- The emphasis in this outcome is on algebraic methods of solving.
- Exponential and logarithmic equations can be solved by a non-graphical process on some graphing calculators.
- The proving of exponential and logarithmic identities is not intended.

(See examples 9, 10, 11, 12, 13, and 21)

Specific Outcome 2.5

Graph and analyze an exponential function, using technology. [R, T, V]

2.5 Notes:

- Exponential functions could be analyzed without technology as a precursor to the technological approach.
- Analysis should include domain, range, x- and y-intercepts, and horizontal asymptotes.
- The relationship between the graph of the exponential function and the graph of its inverse (the logarithmic function) should be introduced here.
- Outcomes from ***Transformations*** should be applied to exponential functions.
- Students should be able to predict the changes to the graph for changing values of b using the form $y = ab^x$.
- Students should be able to use a graphical approach when solving exponential equations.

(See examples 14 and 15)

Specific Outcome 2.6

Model, graph, and apply exponential functions to solve problems. [PS, T, V]

2.6 Notes:

- Care must be taken in solving exponential equations because students may need to have knowledge of logarithmic laws.
- Students should use the context of the question to decide if the data is discrete or continuous. Compound interest is considered discrete where $n \in N$, whereas bacterial growth or decay is continuous where $t \geq 0,\ t \in R$.
- Formulas will be given unless the question fits the form

  $$\text{Final} = \text{Initial (Factor)}^{\frac{\text{total time}}{\text{time for 1 period}}}.$$

- Students are expected to be able to determine the parameters a and b in the exponential function $y = ab^x$ by using the regression feature on their calculator for a given set of data.

(See examples 16, 17, and 18)

Specific Outcome 2.7

Change functions from exponential form to logarithmic form and vice versa. [CN]

2.7 Note:

- Students must be familiar with the laws of exponents.

(See examples 19, 20, and 21)

Specific Outcome 2.8

Use logarithms to model practical problems. [CN, PS, V]

2.8 Note:

- Formulas will be given for problems involving decibels, earthquake intensity, etc.

(See example 22)

Specific Outcome 2.9

Explain the relationship between the laws of logarithms and the laws of exponents. [C, T]

2.9 Note:

- The laws of logarithms could be derived in terms of the laws of exponents.

Specific Outcome 2.10

Graph and analyze logarithmic functions with and without technology. [R, T, V]

2.10 Note:

- Use technology to explore the shape of an exponential function graph. From it, sketch the graph of the inverse (logarithmic) function without technology.

(See example 23)

Acceptable Standard

The student can

- generate the terms of a geometric sequence
- find the general term of a geometric sequence
- write the terms of a geometric series given sigma notation
- find the sum of any finite geometric sequence
- find the sum of a geometric series given the terms or given a general term
- solve for a, r, or n given specified terms and/or other necessary information
- solve for a specified term of a geometric sequence and solve for an exponent in an exponential equation by considering tables, graphs, and common bases
- use a graphing calculator or computer to verify solutions
- solve exponential equations in which the bases are powers of one another
- solve exponential equations in which the bases are **not** powers of one another and the exponents are monomials
- solve a variety of problems by using exponential and/or logarithmic methods
- predict the effect on the graph of an exponential function that has had its base value altered
- identify domain, range, *x*-intercepts, *y*-intercepts, and asymptotes from the graph or from the equation of an exponential function
- graph an exponential function by using technology
- show the relationship between the graph of an exponential function and the graph of its inverse, which is a reflection in the line $y = x$
- understand the relationship between an exponential function and its inverse, and determine what properties of the function are different in the inverse

Standard of Excellence

The student can also

- use sigma notation to represent the sum of a geometric series
- solve for the number of terms in a geometric series, given the sum and other necessary information
- solve exponential equations in which the bases are **not** powers of one another and the exponents are not monomials, or there is a coefficient in front of the base

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• find solutions to exponential function problems when formulas are given	• find solutions to exponential function problems when formulas are not given and must be created algebraically
• use regression methods on a calculator to determine an exponential equation	
• change forms between $y = b^x$ and $\log_b y = x$	• change forms between $y = ab^x$ and $\log_b \frac{y}{a} = x$
• compare values, such as earthquake intensities, given a relation including exponents	• solve for a value or exponent in comparison problems
• explain the relationship between laws of logarithms and laws of exponents	
• use appropriate laws to simplify exponential and logarithmic expressions, and to verify identities	
• identify domain, range, *x*- and *y*-intercepts, and asymptotes from the graph of a logarithmic function of the form $y = \log_b x$	
• graph logarithmic functions with a base other than 10 by using technology	
• solve exponential and logarithmic equations but may not recognize extraneous solutions	• recognize extraneous solutions when solving exponential or logarithmic equations
• participate in and contribute toward the problem-solving process for problems that can be represented by logarithms or exponential functions studied in Pure Mathematics 30	• complete the solutions to problems that can be represented by logarithmic or exponential functions studied in Pure Mathematics 30
• sketch the graph or find the equation when given a single transformation	• sketch the graph or find the equation when given multiple transformations

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *standard of excellence*.

1. What is the seventh term in the geometric series 300 + 60 + 12+ … ?

Solution: $\mathbf{\frac{12}{625}}$

$$
\begin{aligned}
t_7 &= 300\left(\frac{1}{5}\right)^{7-1} \\
&= \frac{12}{625}
\end{aligned}
$$

2. The sum of the first 10 terms of the geometric sequence $-4, 6, -9, \ldots,$ to the nearest tenth, is

A. 153.8
* **B.** 90.7
C. −61.5
D. −453.3

Solution:

$$a = -4, \quad r = -\frac{3}{2}, \quad n = 10$$

$$
\begin{aligned}
S_{10} &= \frac{-4\left(\left(-\frac{3}{2}\right)^{10} - 1\right)}{-\frac{3}{2} - 1} \\
&\doteq 1.6(56.665\ldots) \\
&\doteq 90.66\ldots
\end{aligned}
$$

3. A clothing store is going out of business. The owner reduces the cost of each item by 10% of the current price at the start of each week. A jacket costs \$120.00 during the 1st week of the sale. If this jacket is still in the store during the 5th week of the sale, then the price of the jacket, to the nearest cent, will be

A. \$70.00
B. \$70.86
* **C.** \$78.73
D. \$80.00

Solution:

Sequence: \$120.00, \$108.00, \$97.20, …

$a = 120,\ r = 0.9,\ n = 5$

$$t_5 = 120(0.9)^{5-1} \doteq \$78.73$$

4. Write the first four terms in the series defined as $\sum_{i=2}^{20}[(-1)^{i-1}(2)^{-2i+5}]$.

Solution:

For the first term, $i = 2$ $\quad (-1)^{2-1}(2)^{-2(2)+5} = -2$

For the second term, $i = 3$ $\quad (-1)^{3-1}(2)^{-2(3)+5} = \frac{1}{2}$

For the third term, $i = 4$ $\quad (-1)^{4-1}(2)^{-2(4)+5} = -\frac{1}{8}$

For the fourth term, $i = 5$ $\quad (-1)^{5-1}(2)^{-2(5)+5} = \frac{1}{32}$

SE **5.** Write the geometric series $3 - 6 + 12 - 24 + \ldots + 192$ using sigma notation.

Solution:

$r = -2,\ a = 3,\ t_n = 3(-2)^{n-1}$

$$\begin{aligned} 192 &= 3(-2)^{n-1} \\ 64 &= (-2)^{n-1} \\ (-2)^6 &= (-2)^{n-1} \\ 6 &= n - 1 \\ 7 &= n \end{aligned}$$

Therefore, $\sum_{n=1}^{7} 3(-2)^{n-1}$ represents the geometric series.

Use the following information to answer the next question.

The table below shows the population of Allendale over a 5-year period.

Year	1998	1999	2000	2001	2002	2003
Population	75	150	300	600	1 200	2 400

Assume that the present rate of growth continues.

6. **a.** Determine the equation that models population as an exponential function of the number of years since 1998.

b. Use the equation to determine the predicted population in 2008.

c. The yearly population figures for Allendale form a geometric sequence. Which term of the sequence would yield the predicted population for 2008?

d. Use the formula $t_n = ar^{n-1}$, to find the predicted population for Allendale in 2008.

Solutions:

a. $y = 75(2)^x$, where x is the number of years since 1998

b. Let $x = 10$.
The predicted population is 76 800.

c. The 11th term.

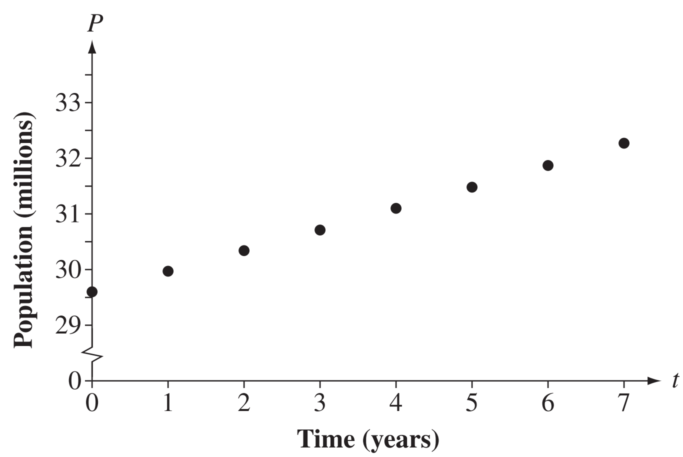
d.

$$\begin{aligned} t_n &= ar^{n-1} \\ t_{11} &= (75)(2)^{11-1} \\ &= (75)(2)^{10} \\ &= 76\ 800 \text{ people} \end{aligned}$$

Use the following information to answer the next question.

In 1995, Canada's population was 29.60 million and was growing at about 1.24% per year. Based on this information, a teacher calculated the predicted population for Canada for seven subsequent years. The teacher then presented his students with the table and graph shown below.

Year	**Time (t) in Years**	**Population (P) in Millions**
1995	0	29.60
1996	1	29.97
1997	2	30.34
1998	3	30.71
1999	4	31.10
2000	5	31.48
2001	6	31.87
2002	7	32.27



7. **a.** Student A wanted to represent the population data as an exponential regression equation in the form $P = a(b)^t$, where t = time in years and P = population in millions. Enter the data in Time(t) as list L_1 and Population(P) as list L_2, and determine the exponential regression values of a, to the nearest hundredth, and b, to the nearest ten thousandth.

b. Student B realized that the data could be modelled as a geometric sequence. The student listed the first three terms of the sequence as $t_1 = 29.60$, $t_2 = 29.97$, and $t_3 = 30.34$, and then wrote the general geometric sequence formula $t_n = 29.60(1.0124)^{n-1}$, $n \in N$.

- Predict the population of Canada, in millions, in the year 2010 using both students' equation models. Explain what the values of t and n represent.
- Which student's equation model is more appropriate to describe Canada's population growth? Support your answer with appropriate explanations.

Solutions:

a. $P = (29.60)(1.0124)^t$

b.

Student A	Student B
• $P(15) = 29.60(1.0124)^{15}$ $P(15) = 35.61$ million t represents the number of years that 2010 is after the year 1995 ($t = 0$). Therefore, the population in 2010 occurs when $t = 15$.	• $t_{16} = 29.60(1.0124)^{16-1}$ $t_{16} = 35.61$ million n represents the position of the term in the sequence. The first term, $n = 1$, represents the population in 1995. Therefore, the population in 2010 is the 16^{th} term in this sequence, so $n = 16$.
• Student A describes population growth as a continuous function.	• Student B describes population growth as discrete values of the average annual growth.

Student A's equation model is more appropriate since population grows throughout each year.

8. Determine the exact value solution for the equation $\left(\frac{1}{8}\right)^{x-3} = 2(16)^{2x+1}$.

Solution: $\mathbf{\frac{4}{11}}$

$$\begin{aligned}
(2^{-3})^{x-3} &= (2)^1(2^4)^{2x+1} \\
2^{-3x+9} &= 2^1(2)^{8x+4} \\
2^{-3x+9} &= 2^{8x+5}
\end{aligned}$$

$\therefore$ by comparing the exponents on the like bases,

$$\begin{aligned}
-3x + 9 &= 8x + 5 \\
4 &= 11x \\
x &= \frac{4}{11}
\end{aligned}$$

9. Verify that $\left(3^{\log x}\right)\left(3^{\log x}\right) = 3^{\log x^2}$, where $x > 0$.

Solution:

If $x = 2$,

Left side: $\left(3^{\log 2}\right)\left(3^{\log 2}\right) = 3^{\log 2 + \log 2}$

$= 3^{\log 4}$

Right side: $3^{\log 2^2} = 3^{\log 4}$

$\therefore$ Left Side = Right Side

SE **10.** An investment of $1 000 is earning 4% interest per annum compounded annually. If the value, V, of the investment after t years is given by $V = 1\,000(1.04)^t$, then t, written as a function of V, is

A. $t = \dfrac{\log_{10}(V)}{3} - \log_{10}(1.04)$

B. $t = \dfrac{\log_{10}(V)}{3\log_{10}(1.04)}$

C. $t = \log_{10}(V) - 3 - \log_{10}(1.04)$

* **D.** $t = \dfrac{\log_{10}(V) - 3}{\log_{10}(1.04)}$

Solution:

$$\begin{aligned}
\log_{10}(V) &= \log_{10}[1\,000(1.04)^t] \\
\log_{10}(V) &= \log_{10} 1\,000 + \log_{10}(1.04)^t \\
\log_{10}(V) &= 3 + t\log_{10}(1.04) \\
t\log_{10}(1.04) &= \log_{10}(V) - 3 \\
t &= \frac{\log_{10}(V) - 3}{\log_{10}(1.04)}
\end{aligned}$$

11. If $\log_3 y = c - \log_3 x$, where $y > 0$ and $x > 0$, then y is equal to

A. $c - x$

B. $\dfrac{c}{x}$

C. $\dfrac{c^3}{x}$

* **D.** $\dfrac{3^c}{x}$

Solution:

$$\begin{aligned}
\log_3 y + \log_3 x &= c \\
\log_3 (y \cdot x) &= c \\
3^c &= y \cdot x \\
y &= \frac{3^c}{x}
\end{aligned}$$

SE **12.** Solve $\log_7(x+1) + \log_7(x-5) = 1$, and verify your solution.

Solution:

$$
\begin{aligned}
\log_7(x+1) + \log_7(x-5) &= 1, \text{ where } x > 5 \\
\log_7[(x+1)(x-5)] &= 1 \\
7^1 &= x^2 - 4x - 5 \\
0 &= x^2 - 4x - 12 \\
0 &= (x-6)(x+2) \\
x - 6 &= 0 \quad \text{or} \quad x + 2 = 0 \\
x &= 6 \quad \text{or} \quad \cancel{x = -2} \text{ extraneous root}
\end{aligned}
$$

Verification:

$$
\begin{aligned}
\log_7(6+1) + \log_7(6-5) &= 1 \\
\log_7 7 + \log_7 1 &= 1 \\
1 + 0 &= 1
\end{aligned}
$$

$$
\begin{aligned}
\log_7(-2+1) + \log_7(-2-5) &= 1 \\
\underbrace{\log_7(-1) + \log_7(-7)}_{\text{impossible}} &= 1
\end{aligned}
$$

impossible $\therefore x = -2$ is an extraneous root

SE **Numerical Response**

13. In the equation $3^{2x+1} = 7$, the value of x, to the nearest hundredth, is ______________.

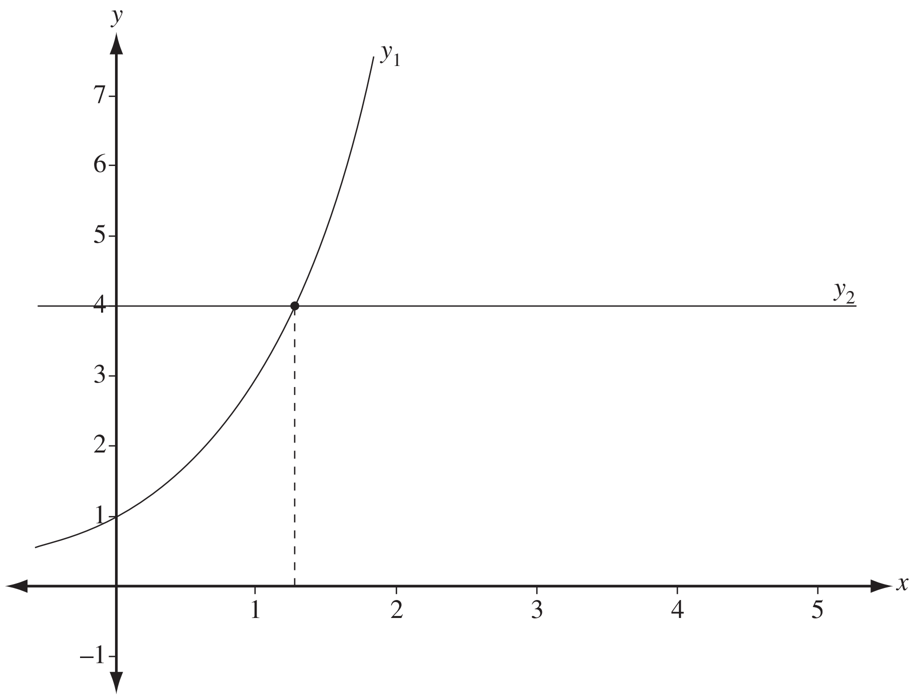
Solution:

$$
\begin{aligned}
\log 3^{2x+1} &= \log 7 \\
(2x+1)\log 3 &= \log 7 \\
2x\log 3 + \log 3 &= \log 7 \\
2x\log 3 &= \log 7 - \log 3 \\
x &= \frac{\log 7 - \log 3}{2\log 3} \quad \text{or} \quad \frac{\log\left(\frac{7}{3}\right)}{\log 9} \\
x &= 0.38562\ldots \\
x &\doteq 0.39
\end{aligned}
$$

14. Use a graphing method to determine the value of x if $3^x = 4$.

Solution:

Use window x: [0, 5, 1] y: [–1, 7, 1]
Graph $y_1 = 3^x$.
Graph $y_2 = 4$.
Determine the x value at the intersection point.
The value of x is approximately 1.26.



Alternate Solution:

Graph $y = 3^x - 4$ and find the x-intercept or real zero. This is the root of the corresponding equation $0 = 3^x - 4$.

Note: In either method, the trace button is not accurate enough to determine the value of x.

15. The graph of $y = f(x) = b^x$, where $b > 1$, is translated such that the equation of the new graph is expressed as $y - 2 = f(x - 1)$. The range of the new function is

* **A.** $y > 2$
B. $y > 3$
C. $y > -1$
D. $y > -2$

Solution:

The range of $y = b^x$ is $y > 0$. The translated graph has been shifted up 2 units, so the new range is $y > 2$.

SE **16.** The population of a city was 173 500 on January 1, 1988, and it was 294 000 on January 1, 2002. If the growth rate of the city can be modelled as an exponential function, then the average annual growth rate of the city, expressed to the nearest tenth of a percent, was

A. 1.0%
* **B.** 3.8%
C. 6.9%
D. 12.1%

Solution:

$$294\,000 = 173\,500(a)^{\frac{14}{1}}, \text{ where } a = 1 + i$$
$$\frac{294\,000}{173\,500} = a^{14}$$
$$a = \left(\frac{294\,000}{173\,500}\right)^{\frac{1}{14}}$$
$$a \doteq 1.038$$

Therefore, the average annual growth rate was approximately 3.8%.

17. The following data represent the cooling of a cup of hot chocolate over time. Use exponential regression to find an equation in the form $T = a(b)^t$, where t = time (min) and T = temperature (°C).

Time (min)	**Temperature (°C)**
0	60
5	54
10	48
15	44
20	41

Solution:

Enter the time data in LIST 1 and temperature data in LIST 2 on a graphing calculator. Perform the exponential regression function on the calculator.
This results in the equation $T \doteq 59.4(0.98)^t$.

Note: For calculators that use the form $y = ae^{(bx)}$ for exponential regression, the resulting equation will be $y = 59.4e^{-0.019t}$. The value of $e^{-0.019}$ is approximately 0.98; therefore, $T \doteq 59.4(0.98)^t$.

SE **Numerical Response**

18. Earthquake intensity is given by $I = I_o(10)^m$, where I_o is the reference intensity and m is magnitude. A particular major earthquake of magnitude 7.9 is 120 times as intense as a particular minor earthquake. The magnitude, to the nearest tenth, of the minor earthquake is ____________.

Solution: 5.8

$$\begin{aligned} I_1 &= I_o(10)^{7.9} \\ \div\ I_2 &= I_o(10)^m \end{aligned}$$

$$\begin{aligned} \frac{I_1}{I_2} &= 10^{(7.9-m)} \\ 120 &= 10^{(7.9-m)} \\ \log_{10}120 &= \log_{10}10^{(7.9-m)} \\ \log_{10}120 &= (7.9-m)\log_{10}10 \\ \log_{10}120 &= 7.9\log_{10}10 - m\log_{10}10 \\ m &= 7.9 - \log_{10}120 \\ m &\doteq 5.8 \end{aligned}$$

19. If $\log_x\left(\frac{1}{64}\right) = -\frac{3}{2}$, then x is equal to

* **A.** 16

B. 8

C. $\frac{1}{8}$

D. $\frac{1}{16}$

Solution:

$$x^{\frac{-3}{2}} = \frac{1}{64}$$

$$\left(x^{\frac{-3}{2}}\right)^{\frac{-2}{3}} = \left(\frac{1}{64}\right)^{\frac{-2}{3}}$$

$$x = 16$$

20. The equation $y = 4^{3x}$ can also be written as

A. $y = \frac{\log_3 x}{4}$

B. $y = \frac{\log_4 x}{3}$

C. $x = \frac{\log_3 y}{4}$

* **D.** $x = \frac{\log_4 y}{3}$

Solution:

If $c = a^b$, then $b = \log_a c$.

Given $y = 4^{3x}$, then $3x = \log_4 y$

$$x = \frac{\log_4 y}{3}$$

or

$$\log y = \log(4^{3x})$$

$$\log y = 3x \cdot \log 4$$

$$\frac{\log y}{\log 4} = 3x$$

$$\log_4 y = 3x$$

$$\frac{\log_4 y}{3} = x$$

SE **21.** If $\log_3 x = 15$, then $\log_3\left(\frac{1}{3}x\right)$ is equal to

* **A.** 14
B. 12
C. 5
D. –15

Solution:

$$\begin{aligned} \log_3 x &= 15 \\ x &= 3^{15} \\ \therefore \quad \log_3\left(\frac{1}{3}x\right) &= \log_3\left(\frac{1}{3}\cdot 3^{15}\right) \\ &= \log_3(3^{14}) \\ &= 14 \end{aligned}$$

Use the following information to answer the next question.

An equation that defines the decibel level for any sound is

$L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where L = loudness in decibels
I = intensity of sound being measured
I_0 = intensity of sound at the threshold of hearing

22. Given that normal conversation is 1 000 000 times as intense as I_0, the loudness of normal conversation is

A. 5 decibels
B. 6 decibels
C. 16 decibels
* **D.** 60 decibels

Solution:

$$\begin{aligned} L &= 10\log_{10}\left(\frac{1\,000\,000 I_0}{I_0}\right) \\ &= 10\log_{10}(1\,000\,000) \\ &= 10\cdot 6 \\ &= 60 \end{aligned}$$

23. Graph $y = \log_2 x$, and identify the domain, range, x- and y-intercepts, and asymptotes.

Solution:

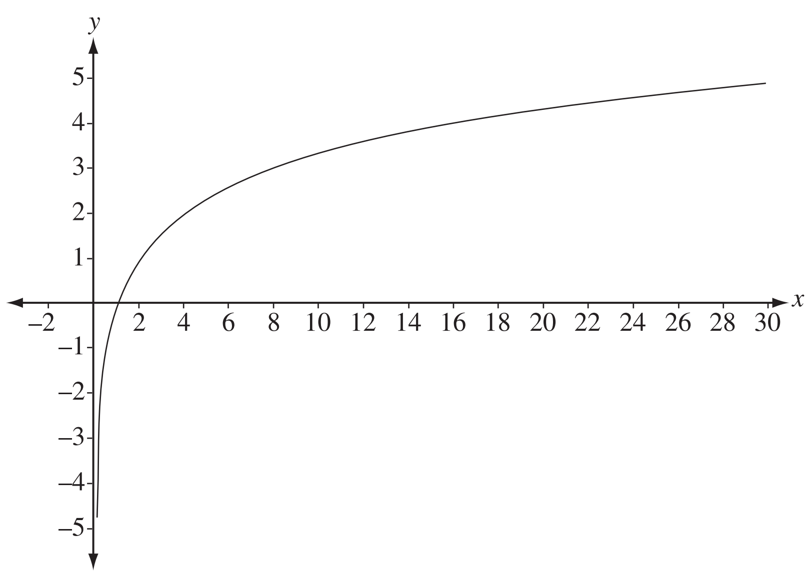
$$y = \log_2 x$$
$$2^y = x$$

x	$\frac{1}{2}$	1	2	4	8
y	–1	0	1	2	3

Domain: $x > 0$
Range: $y \in R$
Intercepts x: (1, 0)
y: none
Asymptote: $x = 0$ or y-axis

or

Graph $y = \dfrac{\log_{10} x}{\log_{10} 2}$ using a graphing calculator.



Trigonometry

General Outcome

Solve trigonometric equations and identities.

Represent and analyze trigonometric functions, using technology as appropriate.

General Notes:

- The window format for graphing calculators is x: [x_{min}, x_{max}, x_{scl}] y: [y_{min}, y_{max}, y_{scl}].
- Students should be familiar with graphing in radian mode and degree mode.
- Students should be made aware of the difference between exact values and approximate values.
- When a graphical solution is used in solving an equation or a system of equations, students should include the function or functions graphed, the window settings used, a sketch of the resulting calculator display, and a clear explanation of how the solution is approximated. (See *example 9a*)

Specific Outcomes

Specific Outcome 3.1

Distinguish between degree and radian measure, and solve problems using both. [CN, E]

3.1 Notes:

- Students need to be aware that their calculators must be in the correct mode (degree or radian).
- When cleared, graphing calculators default to **radian** mode.
- Students must carry forward the numerical accuracy displayed on their calculators through all parts of the question. **Rounding must not occur until the final solution is determined.**
- Solving problems involving arc length, radius, and angle measure in either radians or degrees should be included.
- Some calculators have built-in functions to convert from radians to degrees, and these may be used.

(See examples 1, 2, and 3)

Specific Outcome 3.2

Determine the exact and the approximate values of trigonometric ratios for any multiples of $0°$, $30°$, $45°$, $60°$, and $90°$, and 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. [CN, E]

3.2 Notes:

- These angles are referred to as special angles.
- The reciprocal ratios should be introduced at this time.
- These questions can be done as exact values on graphing calculators that have symbolic manipulation capabilities. They will not be assessed as individual questions on the diploma examination, but as parts of more involved questions.
- Special triangles are not covered within the specific outcomes of the Pure Mathematics 10 or 20 programs.

(See examples 4, 5, 6, and 7)

Specific Outcome 3.3

Solve first and second degree trigonometric equations over the domain $0 \le \theta < 2\pi$ or the domain $0° \le \theta < 360°$

- algebraically
- graphically

[PS, T]

3.3 Notes:

- Students with graphing calculators that have symbolic manipulation capabilities can solve these problems through the "solve" feature, either in approximate or exact form. On the diploma examination, these questions will be a part of a more involved question.
- Restrict reciprocal function equations to first degree only.
- Solutions to equations can only be approximated from the graph, therefore using the trace button on your calculator may not be accurate enough for a numerical-response question.
- Restrict equations involving multiple angles, $k\theta$, to $k = \frac{1}{3}, \frac{1}{2}, 2, 3$.
- Students are expected to be able to use trigonometric identities in the solution of trigonometric equations.

(See examples 8, 9, 10, 11, 12, 13, 14d, 15, and 16)

Specific Outcome 3.4

Determine the general solutions to trigonometric equations where the domain is the set of real numbers. [PS, T]

3.4 Notes:

- The student should be able to write an expression for the general solution to a trigonometric equation.
- The student should be able to solve trigonometric equations over the domain of the real numbers.
- Some of the examples or questions could be done with graphing calculators that have symbolic manipulation capabilities by entering statements such as Solve ($x - 2 \sin x = 0, x$).

(See examples 14e and 17)

Specific Outcome 3.5

Verify trigonometric identities

- numerically for any particular case
- algebraically for general cases
- graphically

[PS, R, T, V]

3.5 Notes:

- Trigonometric functions may have either point or asymptote discontinuities that must be addressed in the analysis of the functions.
- In terms of the *Mathematics and Science Directing Words* (pages 128 and 129), students must *verify* numerically, *prove* algebraically, and *verify* graphically trigonometric identities appropriate to the Pure Mathematics 30 level.
- Students may verify identities numerically by using either approximate or exact values.

(See examples 14, 18, 19, and 20)

Specific Outcome 3.6

Describe sine, cosine, and tangent as circular functions, with reference to the unit circle and an angle in standard position. [PS, R, V]

3.6 Notes:

- This outcome should also be understood for the three reciprocal ratios.
- The trigonometric functions of θ should be defined in terms of x, y, and r for all points on the coordinate plane (including quadrants III and IV).
- This concept may be developed by using the unit circle, special triangles, symmetry, etc.

(See examples 21, 22, and 23)

Specific Outcome 3.7

Use sum, difference, and double-angle identities for sine and cosine to verify and simplify trigonometric expressions. [R, T]

3.7 Notes:

- Students should be able to use the sum identities to prove, derive, and simplify the double-angle identities of $\sin(2x)$ and $\cos(2x)$.
- Students should be able to express the $\cos(2x)$ identity in three different ways.

(See examples 9b, 24, and 25)

Specific Outcome 3.8

Draw (using technology), sketch, and analyze the graphs of sine, cosine, and tangent functions for

- amplitude, if defined
- period
- domain and range
- asymptotes, if any
- behaviour under transformations

[CN, T, V]

3.8 Notes:

- Behaviour of functions under transformations includes horizontal phase shift and vertical displacement.
- Transformations that are vertical and horizontal stretches can be described by using terms such as amplitude and period.
- Sinusoidal functions will be expressed in the form $y = a \sin[b(x - c)] + d$ or $y = a \cos[b(x - c)] + d$. If students choose to use regression models on graphing calculators, they have the responsibility of adjusting to this form.
- Discuss how the values of a, b, c, and d affect the characteristics of the graph for sine, cosine, and tangent functions only.
- Sinusoidal regression is not an expected outcome, but technology may be used to find the equation of a sinusoidal curve.
- It is not intended that inverse trigonometric functions (e.g., $y = \sin^{-1}x$ or $y = \text{arc} \sin x$) be studied in this course. They are to be used only in calculations.

(See examples 15, 26, and 27)

Specific Outcome 3.9

Draw and sketch (using technology) and analyze the graphs of secant, cosecant, and cotangent functions, for

- period
- domain and range
- asymptotes
- behaviour under horizontal and vertical stretches

[CN, T, V]

3.9 Notes:

- If $f(x) = \sin x$, the reciprocal function is $\csc x$ and must not be denoted by $\sin^{-1}x$. $\left(\text{i.e., } \dfrac{1}{f(x)} \text{ is not the same as } f^{-1}(x)\right)$
- Only behaviour under stretching transformations, i.e., changing parameters a and b, should be discussed.
- The importance of selecting the correct window on a graphing calculator should be discussed.

(See examples 28 and 29)

Specific Outcome 3.10

Use sine and cosine functions to model and solve problems. [PS, R, V]

3.10 Notes:

- In modelling problems based on trigonometric functions, students should be able to recognize that the horizontal axis can be measured in units other than angle measurement, e.g., time, distance.
- Students should be able to distinguish between discrete and continuous data.

(See examples 30 and 31)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• convert from degrees to radians and vice versa	• apply the concepts of arc length to more difficult multistep real-world applications
• solve problems involving arc length, radius, and angle measure in either radians or degrees	
• find exact values of trigonometric ratios for special angles, θ, where $0 \le \theta < 2\pi$ or $0° \le \theta < 360°$	• find exact values of trigonometric ratios for any integral multiples of the special angles, θ, where $\theta < 0°$ or 0 rad, or where $\theta \ge 360°$ or $\theta \ge 2\pi$ rad.
• explain why certain trigonometric ratios can have undefined values	
• identify restrictions on the variable in the domain $0 \le \theta < 2\pi$	• identify restrictions on the variable in the domain $\theta \in R$
• in a given domain, algebraically and graphically determine – all solutions to first-degree equations – partial solutions to second-degree equations – partial solutions to trigonometric equations involving reciprocal trigonometric ratios – partial solutions to trigonometric equations involving multiple angles, $k\theta$, where $k = \frac{1}{3}, \frac{1}{2}, 2, 3$	• in a given domain, algebraically and graphically determine – all solutions to second-degree equations – all solutions to trigonometric equations involving reciprocal trigonometric ratios – all solutions to trigonometric equations involving multiple angles, $k\theta$, where $k = \frac{1}{3}, \frac{1}{2}, 2, 3$
	• determine the general solution of trigonometric equations
• verify identities for a particular case	
• simplify and prove simple identities algebraically for a general case	• simplify and prove more difficult identities algebraically for the general case, such as those requiring the use of conjugates or extensive use of rational operations
• graph both sides of an identity to verify it	
• find exact values for any of the six trigonometric ratios, given a point on the terminal arm of the angle in standard position	
• find any other trigonometric ratios given one ratio and a quadrant specification	• find other trigonometric ratios given one ratio and no quadrant specification
• find an angle in standard position such that its terminal arm passes through a given point	
• verify sum and difference and double-angle identities of sine and cosine for a particular case	

Acceptable Standard	*Standard of Excellence*
• algebraically prove identities involving sum and difference and double-angle identities	• algebraically prove more difficult identities involving sum and difference and double-angle identities
• simplify expressions by using sum and difference and double-angled identities of sine and cosine	• identify the values of a variable for which an identity is undefined over the domain $0 \le \theta < 2\pi$ or $0 \le x < 360°$
• create a graph using technology for primary trigonometric functions and analyze the graph for features such as domain, range, amplitude, period, and asymptotes	
• give partial explanations of the relationships between equation parameters and graphs of primary trigonometric functions	• give full explanations of the relationships between equation parameters and transformations of primary trigonometric functions
• create a graph using technology for reciprocal trigonometric functions	• analyze the graphs of reciprocal trigonometric functions for domain, range, period, and asymptotes
• perform single transformations involving horizontal or vertical stretches of reciprocal trigonometric functions	• perform both a horizontal and vertical stretch on a reciprocal trigonometric function
• graph a sinusoidal curve to model a problem, given the equation	
• develop the partial equation for a sinusoidal curve, given the graph	• develop the complete equation of a sinusoidal curve, given either a description of a real-world problem or the graph
• create a partial graph of a sinusoidal curve, given a description of a real-world application	• create a complete graph of a sinusoidal curve, given a description of a real-world application
• participate in and contribute toward the problem-solving process for problems that require the analysis of trigonometry studied in Pure Mathematics 30	• complete the solution to problems that require the analysis of trigonometry studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *standard of excellence*.

1. **a.** Convert the angles $\frac{2\pi}{3}$ and 1.6 rad to degrees, correct to the nearest tenth.

b. Convert the angle 175° to radians as an exact value and an approximate value correct to four decimal places.

Solutions:

a. $\frac{2\pi}{3} = 120.0°$

$1.6 \text{ rad} \doteq 91.7°$

b. $175° \times \frac{\pi}{180°} = \frac{35\pi}{36}$ (exact) or 3.0543 rad (approximate)

2. In one minute, the second hand of a clock completes one revolution around the clock face. In $1\frac{1}{2}$ minutes, the second hand of a clock completes an angle of

A. $\frac{3\pi}{2}$

* **B.** 3π

C. 6π

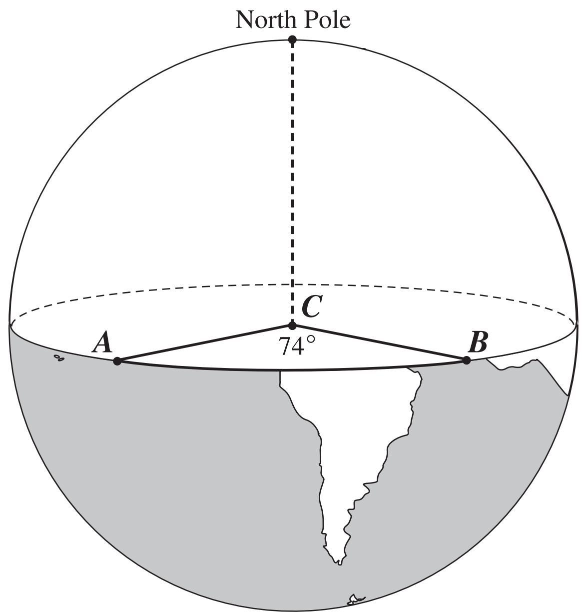
D. 180π

Solution:

$1\frac{1}{2} \times 2\pi = 3\pi$

Use the following information to answer the next question.

Two points, A and B, are on Earth's equator, and point C is at the centre of Earth. The measure of $\angle ACB$ is $74°$, as shown below.



SE **3.** If the circumference of Earth at the equator is approximately 40 070 km, then the shortest arc length from point A to point B, correct to the nearest kilometre, is

A. 31 026 km
B. 16 474 km
***C.** 8 237 km
D. 4 938 km

Solution:

$$74° = \frac{74\pi}{180} \text{ rad}$$

$$\text{then } \frac{\frac{74\pi}{180}}{2\pi} = \frac{a}{40\,070 \text{ km}}$$

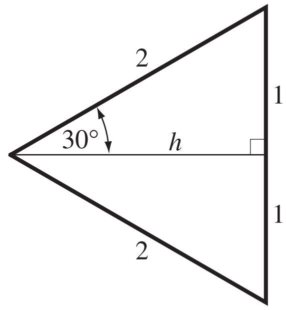
$$a \doteq 8\,237 \text{ km}$$

or

$$C = 2\pi r$$
$$40\,070 = 2\pi r$$
$$\frac{40\,070}{2\pi} = r$$
$$r \doteq 6\,377.3386$$
$$\theta = 74° \text{ or } 1.29154 \text{ rad}$$

$$a = r\theta$$
$$= (6\,377.3386)(1.29154)$$
$$= 8\,236.5879$$
$$a \doteq 8\,237 \text{ km}$$

4. Given an equilateral triangle with a side of 2 units, determine the exact primary trigonometric ratios of 30°.



Solutions:

$$\begin{aligned} 2^2 &= 1^2 + h^2 \\ 3 &= h^2 \\ \sqrt{3} &= h \end{aligned}$$

$$\cos 30° = \frac{\sqrt{3}}{2},\ \sin 30° = \frac{1}{2},\ \tan 30° = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}$$

5. **a.** Explain why $\tan\left(\frac{\pi}{2}\right)$ is undefined.

b. Are there any of the remaining five trigonometric ratios that have undefined values? Explain why.

Solution:

a. $\tan A = \frac{y}{x}$; therefore, when $x = 0$, $\frac{y}{0}$ is undefined. This occurs at $A = \frac{\pi}{2}$.

b. Yes; csc A, sec A, and cot A all have denominators of x or y, and will be undefined when their denominators equal zero.

6. Determine the exact value of $\sin^2\left(\frac{7\pi}{6}\right) + \tan^2\left(\frac{\pi}{3}\right)$.

Solution:

$$\left[\sin\left(\frac{7\pi}{6}\right)\right]^2 + \left[\tan\left(\frac{\pi}{3}\right)\right]^2 = \left(-\frac{1}{2}\right)^2 + \left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)^2 = \frac{1}{4} + 3 = 3\frac{1}{4}$$

SE **7.** If $\cos\theta - k = 0$, and $\theta = -120°$, then the value of k is

A. $\frac{1}{2}$

* **B.** $-\frac{1}{2}$

C. $-\frac{\sqrt{3}}{2}$

D. $\frac{\sqrt{3}}{2}$

Solution:

Let $\theta = -120°$,

$$\begin{aligned} \cos(-120°) - k &= 0 \\ -\frac{1}{2} - k &= 0 \\ k &= -\frac{1}{2} \end{aligned}$$

8. **a.** Given that $\sin\theta = 0.5395$ and that θ is an acute angle, what is θ in radians, to the nearest hundredth?

b. Given that $\sin\theta = 0.5395$ and that $0 \le \theta < 2\pi$, are there any additional solutions? If so, find them and explain why they are also solutions.

c. Explain why $\sin\theta = 1.5395$ has no solution for θ.

Solutions:

a. $\theta \doteq 0.57$ radians

b. Yes, since the sine ratio is also positive in quadrant II, θ has two solutions.
$\theta = \pi - 0.57 \doteq 2.57$ radians.

c. Because the range of the graph of $f(\theta) = \sin\theta$ is $-1 \le f(\theta) \le 1$, there are no values of θ, where $\sin\theta > 1$.

or

Because $\sin\theta = \frac{y}{r}$ and it must hold true that $\frac{y}{r} \le 1$.

9. Determine the solutions to the following trigonometric equations.

a. Graphically solve $1 + 2\cos x = 5\cos x$, where $0 \le x < 2\pi$. Give solutions in decimal form, to the nearest hundredth, and explain how the solution was obtained.

b. Algebraically solve $1 + 2\cos x = 5\cos x$, where $0 \le x < 2\pi$. Give solutions in decimal form, correct to the nearest hundredth.

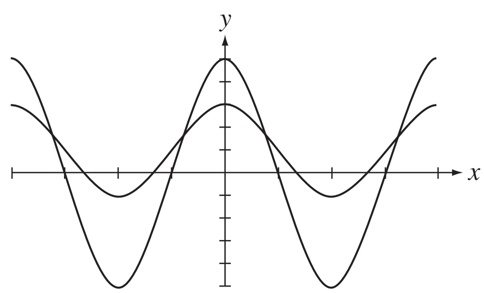
SE **c.** Algebraically solve $\sin(2x) = \cos x$, where $0 \le x < 2\pi$. Give solutions as exact values.

Solutions:

a. Enter the following two functions into the calculator

$$y_1 = 1 + 2\cos x$$
$$y_2 = 5\cos x$$

The window to be used is $x: \left[0, 2\pi, \frac{\pi}{2}\right]$, $y: [-5, 5, 1]$.



Use a Calc Intersect or G•SLV process to determine the x-coordinates of the points of intersection between y_1 and y_2. The x-coordinates are the solutions to the original equation. The solutions are approximately 1.23 rad and 5.05 rad.

b.

$$1 = 5\cos x - 2\cos x$$
$$1 = 3\cos x$$
$$\frac{1}{3} = \cos x$$
$$x \doteq 1.23 \text{ rad}, \ 5.05 \text{ rad}$$

c. $\sin(2x) = \cos x$

$$\begin{aligned} 2\sin x\cos x &= \cos x \\ 2\sin x\cos x - \cos x &= 0 \\ \cos x\,(2\sin x - 1) &= 0 \\ \cos x = 0 \ \textbf{or}\ \sin x &= \frac{1}{2} \\ x = \frac{\pi}{2}, \frac{3\pi}{2}, \ \textbf{or}\ x &= \frac{\pi}{6}, \frac{5\pi}{6}, \\ x &= \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2} \end{aligned}$$

10. When the following pairs of functions are graphed, the pair that could **not** be used to solve the equation $4\sin x - 1 = 0$ is

* **A.** $y = \sin x$ and $y = 1$

B. $y = \sin x$ and $y = \frac{1}{4}$

C. $y = 4\sin x$ and $y = 1$

D. $y = 4\sin x - 1$ and $y = 0$

Solution:

These two functions will yield solutions to $\sin x = 1$ or $\sin x - 1 = 0$ not $4\sin x - 1 = 0$.

11. **a.** Graph $y = 6\sin^2 A - \sin A - 1$, using the window x: $\left[\frac{-\pi}{2}, \frac{5\pi}{2}, \frac{\pi}{2}\right]$, y: $[-2, 6, 1]$ on your graphing calculator.

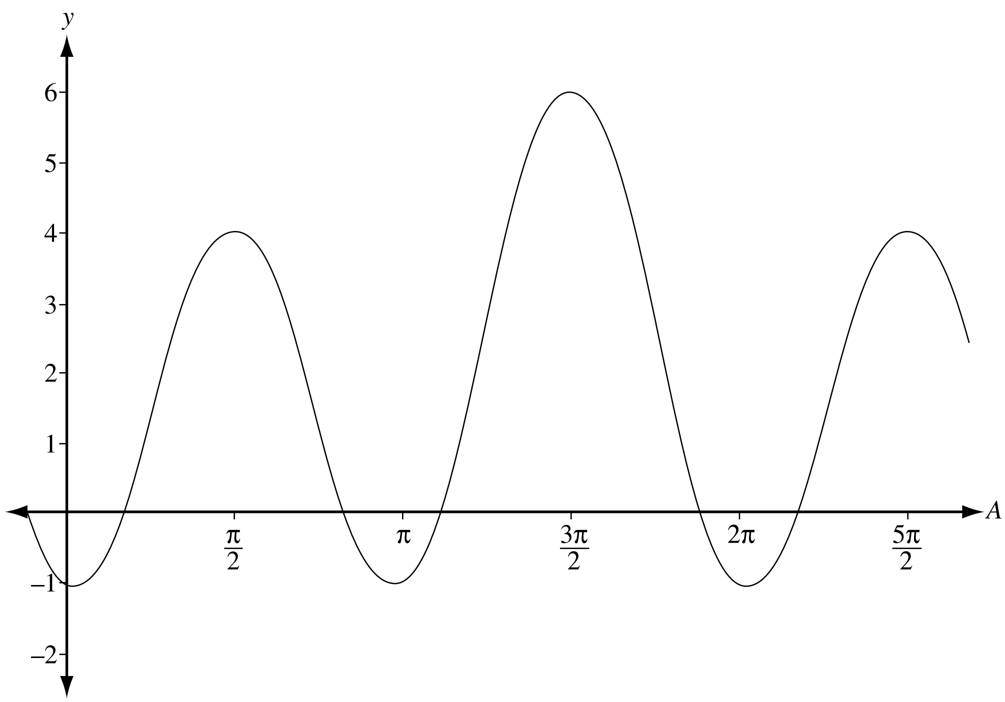
b. Predict the number of solutions to $6\sin^2 A - \sin A - 1 = 0$, $0 \le A < 2\pi$, from your graph and give a reason for your prediction.

c. Using your calculator, estimate the solutions of $6\sin^2 A - \sin A - 1 = 0$, $0 \le A < 2\pi$, to the nearest tenth.

SE **d.** Confirm your estimates algebraically by finding A, correct to the nearest hundredth, for $0 \le A < 2\pi$.

Solutions:

a.



b. Four solutions exist because there are four intercepts between 0 and 2π on the horizontal axis.

c. Use the appropriate calculator function that approximates the *x*-intercepts or zeros to find $A \doteq 0.5, 2.6, 3.5, 5.9$ radians.

d.

$$\begin{aligned} 6\sin^2 A - \sin A - 1 &= 0 \\ (2\sin A - 1)(3\sin A + 1) &= 0 \\ \sin A &= \frac{1}{2}, -\frac{1}{3} \\ A &\doteq 0.52,\ 2.62,\ 3.48,\ 5.94 \text{ radians} \end{aligned}$$

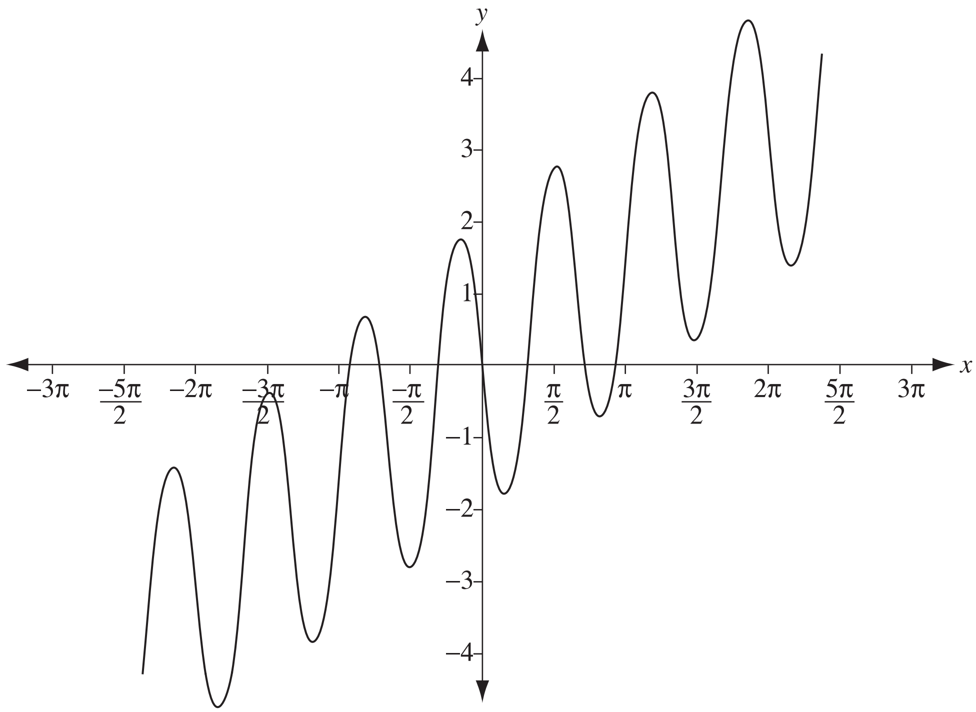
12. Use technology to graph $y = \frac{x}{2} - 2\sin(3x)$ in radian mode, and then

a. state the total number of *x*-intercepts over the real numbers

b. find the lowest possible value of *x*, to the nearest tenth, for $\frac{x}{2} - 2\sin(3x) = 0$

c. explain how a student could answer part a above by entering $y_1 = \frac{x}{2}$ and $y_2 = 2\sin(3x)$ on a graphing calculator

Solutions:



a. 7

b. Since y has been replaced by 0, the lowest possible value of x which satisfies the equation is the x-intercept furthest to the left. Using the appropriate calculator function, $x \doteq -2.9$.

c. The student could count the number of points of intersection between the two graphs. The x-coordinates of these points would be the x-intercepts of the original function.

13. Given that x is an acute angle, determine x, correct to the nearest degree, such that $\sin x \cot x = 0.4$.

Solution:

$$\begin{aligned} \sin x \cot x &= 0.4 \\ \sin x\left(\frac{\cos x}{\sin x}\right) &= 0.4 \\ \cos x &= 0.4 \\ x &\doteq 66^\circ \end{aligned}$$

14. Given the identity $\dfrac{\sin x}{1 - \cos x} = \dfrac{1 + \cos x}{\sin x}$, where $\cos x \neq 1$ and $\sin x \neq 0$,

a. verify the identity for the particular case $x = \dfrac{\pi}{3}$

b. demonstrate by graphing $y_1 = \dfrac{\sin x}{1 - \cos x}$ and $y_2 = \dfrac{1 + \cos x}{\sin x}$ that the graphs verify the identity

c. identify the values of x for which this identity is undefined over the domain $0 \leq x < 2\pi$

SE **d.** identify the values of x for which this identity is undefined

Solutions:

a.

Left Side	Right Side
$\dfrac{\sin \frac{\pi}{3}}{1 - \cos \frac{\pi}{3}} = \dfrac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$	$\dfrac{1 + \cos \frac{\pi}{3}}{\sin \frac{\pi}{3}} = \dfrac{1 + \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{3}{\sqrt{3}} = \sqrt{3}$

b. The equations $y_1 = \dfrac{\sin x}{1 - \cos x}$ and $y_2 = \dfrac{1 + \cos x}{\sin x}$ appear to produce identical graphs for the window setting. Thus, $\dfrac{\sin x}{1 - \cos x} = \dfrac{1 + \cos x}{\sin x}$ for all x is a reasonable conjecture.

Note: However, points of discontinuity on the graph of y_2 occur at $x = (2n + 1)\pi,\ n \in I$. It may be difficult to see these points of discontinuity on the calculator screen; however, inputting $x = \pi, -\pi$, etc., into the table features of a graphing calculator will show an error value for y.

c. $x = 0$ and $x = \pi$

d. $\dfrac{\sin x}{1 - \cos x}$ is undefined when $1 - \cos x = 0$; therefore, when $x = 2n\pi,\ n \in I$, the expression is undefined

$\dfrac{1 + \cos x}{\sin x}$ is undefined when $\sin x = 0$; therefore, when $x = n\pi,\ n \in I$, the expression is undefined

$\therefore$ the identity is undefined when $x = n\pi,\ n \in I$

15. The three solutions of the equation $f(\theta) = 0$ are $0°$, $180°$, and $360°$. Therefore, the three solutions of the equation $f(\theta - 30°) = 0$ are

A. $0°$, $60°$, and $120°$
B. $-30°$, $150°$, and $330°$
* **C.** $30°$, $210°$, and $390°$
D. $0°$, $540°$, and $1\,080°$

Solution:

The graph of $y = f(\theta - 30°)$ is a horizontal (phase) shift of 30° right of the graph of $y = f(\theta)$. Therefore, the zeros or solutions to $f(\theta) = 0$ will also be shifted 30° to the right to become the zeros or solution to $f(\theta - 30°) = 0$.

16. Algebraically determine the solution to the following equations.

a. $\sin(2x) = \frac{-\sqrt{3}}{2}$, $0° \le x < 360°$

b. $\cot(x) = 1.4$, $0 \le x < 2\pi$, correct to the nearest hundredth of a radian

Solutions:

a. $2x = 240°, 300°, 600°, 660°$, $0° \le 2x < 720°$
$x = 120°, 150°, 300°, 330°$

b. $x \doteq 0.62$ rad, 3.76 rad

17. Determine the general solution of $(2 \sin A - 1)(\tan A - 1) = 0$.

Solution:

$\sin A = \frac{1}{2}$ or $\tan A = 1$

$A = \frac{\pi}{6} + 2n\pi, \ \frac{5\pi}{6} + 2n\pi, \ \frac{\pi}{4} + n\pi, \ n \in I$

SE Students who achieve the *acceptable standard* can find partial answers (e.g., only give one solution for one rotation). Students who achieve the *standard of excellence* can write an expression for the general solution.

18. To create an identity (a statement that is true for all values of x in the domain) for the equation $\cos^2 x(1 + \cot^2 x) = A$, the value of A would need to be

A. $\sin^2 x$
B. $\cos^2 x$
* **C.** $\cot^2 x$
D. $\sec^2 x$

Solution:

$$\cos^2 x(1+\cot^2 x)$$
$$\cos^2 x(\csc^2 x)$$
$$\cos^2 x\left(\frac{1}{\sin^2 x}\right)$$
$$\frac{\cos^2 x}{\sin^2 x}$$
$$\cot^2 x$$

SE **19.** The expression $\dfrac{\sin x+\cos x}{\csc x+\sec x}$ is equivalent to

* **A.** $\sin x \cos x$
B. $\tan^2 x \sin x$
C. $\sec x \csc x$
D. $\sin^2 x + \cos^2 x$

Solution:

This is an example of a "more difficult identity."

$$\begin{aligned}\frac{\sin x + \cos x}{\dfrac{1}{\sin x} + \dfrac{1}{\cos x}} &= \frac{\sin x + \cos x}{\dfrac{\cos x + \sin x}{\sin x \cos x}} \\ &= (\sin x + \cos x) \times \frac{\sin x \cos x}{\cos x + \sin x} \\ &= \sin x \cos x\end{aligned}$$

20. Prove algebraically that $\sin x \cot x = \cos x$, $\sin x \neq 0$.

Solution:

$$\begin{aligned} \sin x \cot x &= \cos x \\ \sin x\left(\frac{\cos x}{\sin x}\right) &= \cos x \\ \cos x &= \cos x \end{aligned}$$

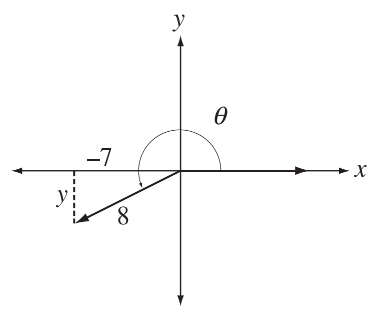
21. Given that $\cos\theta = -\frac{7}{8}$ and $\tan\theta > 0$, determine the exact value of $\sin\theta$.

Solution:

Since $\cos\theta < 0$ and $\tan\theta > 0$, θ is a third quadrant angle.

$$\begin{aligned} y^2 + (-7)^2 &= 8^2 \\ y^2 &= 15 \\ y &= -\sqrt{15} \end{aligned}$$

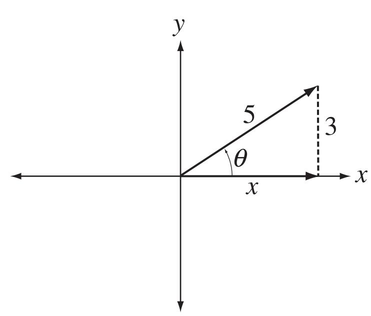
Therefore, $\sin\theta = \dfrac{-\sqrt{15}}{8}$



22. Given that $\csc\theta = \frac{5}{3}$, $0 < \theta < \frac{\pi}{2}$, determine the five other trigonometric ratios.

Solution:

$$\begin{aligned} 3^2 + x^2 &= 5^2 \\ x^2 &= 16 \\ x &= 4 \end{aligned}$$



$$\sin\theta = \frac{3}{5}$$

$$\cos\theta = \frac{4}{5},\ \sec\theta = \frac{5}{4}$$

$$\tan\theta = \frac{3}{4},\ \cot\theta = \frac{4}{3}$$

23. If the terminal arm of angle θ, in standard position, passes through point $(-b, 2b)$, where $b > 0$, then the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ are, respectively,

A. $\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}$, and 2

* **B.** $\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}$, and -2

C. $\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}$, and -2

D. $\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}$, and 2

Solution:

$$\begin{aligned} x &= -b \\ y &= 2b \\ (-b)^2 + (2b)^2 &= r^2 \\ 5b^2 &= r^2 \\ r &= \sqrt{5}b \end{aligned}$$

$$\sin\theta = \frac{y}{r} = \frac{2b}{\sqrt{5}b} = \frac{2}{\sqrt{5}}$$

$$\cos\theta = \frac{x}{r} = \frac{-b}{\sqrt{5}b} = \frac{-1}{\sqrt{5}}$$

$$\tan\theta = \frac{y}{x} = \frac{2b}{-b} = -2$$

24. Given $f(x) = \dfrac{1-\cos^2 x}{\tan x}$,

a. graph $y = f(x)$ using the window settings $x: \left[-2\pi, 2\pi, \dfrac{\pi}{2}\right]$ $y: \left[-1, 1, 0.5\right]$

b. state the period and the amplitude of this function

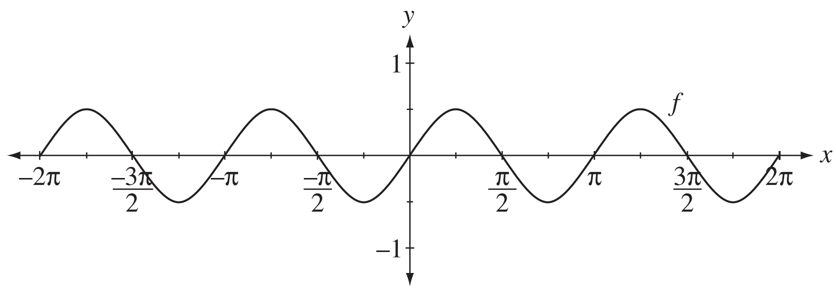
SE **c.** find the values of x for which the function is undefined

d. state a single trigonometric function that is equivalent to $f(x) = \dfrac{1-\cos^2 x}{\tan x}$

(Hint: $y = a\sin(bx)$ and the graph generated in *a*.)

Solutions:

a.



b. Period is π, amplitude is $\dfrac{1}{2}$

c. The function is undefined when $\tan x = 0$ or when $\tan x$ is undefined, therefore, when $x = \dfrac{n\pi}{2}$, $n \in I$, the function is undefined. There are point discontinuities on the graph where $x = \dfrac{n\pi}{2}$, $n \in I$.

d. $y = \dfrac{1}{2}\sin(2x)$

Use the following information to answer the next question.

Refraction describes the bending of light rays. Refraction can be calculated by using the formula

$$n = \frac{\sin(\theta + \alpha)}{\sin \theta},$$

where n represents the refractive index of the material.

25. If $\alpha = 30°$, then an equivalent expression for n is

A. $\frac{\sqrt{3}}{2} + \cos\theta$

* **B.** $\frac{\sqrt{3}}{2} + \frac{1}{2}\cot\theta$

C. $\frac{\sqrt{3}}{2} + \frac{1}{2}\cos\theta$

D. $\frac{1}{2} + \frac{\sqrt{3}}{2}\cot\theta$

Solution:

$$\begin{aligned}
n &= \frac{\sin(\theta + 30°)}{\sin\theta} \\
&= \frac{\sin\theta\cos 30° + \cos\theta\sin 30°}{\sin\theta} \\
&= \frac{\sin\theta\left(\frac{\sqrt{3}}{2}\right) + \cos\theta\left(\frac{1}{2}\right)}{\sin\theta} \\
&= \frac{\frac{\sqrt{3}}{2}\sin\theta}{\sin\theta} + \frac{\frac{1}{2}\cos\theta}{\sin\theta} \\
&= \frac{\sqrt{3}}{2} + \frac{1}{2}\cot\theta
\end{aligned}$$

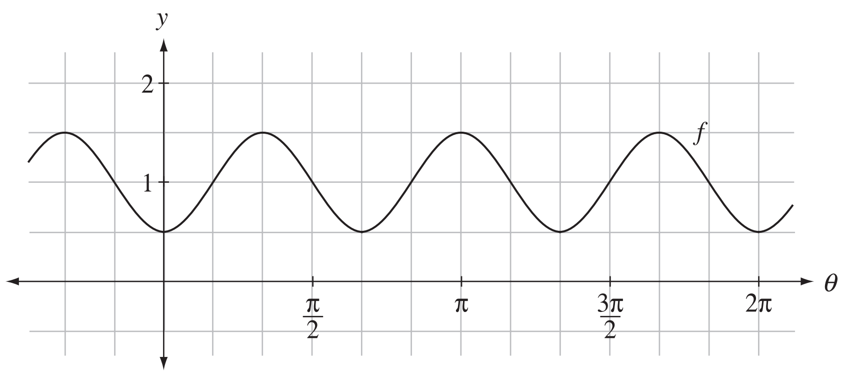
26. Which of the following statements does **not** describe the graph of $f(\theta) = -3\sin\left(\theta - \frac{\pi}{2}\right)$?

A. The amplitude is 3.
B. The period is 2π.
C. This graph is the same as the graph of $f(\theta) = -3\sin(\theta)$ with a phase shift of $\frac{\pi}{2}$ to the right.
* **D.** This graph is the same as the graph of $f(\theta) = \sin\left(\theta - \frac{\pi}{2}\right)$ with a vertical translation of 3 units down.

Solution:

This statement does not describe $f(\theta)$ because the parameter –3 is not a vertical translation down.

Use the following graph to answer the next question.



SE **27.** Given a partial graph of $y = f(\theta)$ with a minimum point at $\left(0, \frac{1}{2}\right)$ and a maximum point at $\left(\frac{\pi}{3}, \frac{3}{2}\right)$,

a. identify the amplitude and period
b. write the function in the form $y = a\sin[b(\theta - c)] + d$
c. write the function in the form $y = a\cos[b(\theta - c)] + d,\ a > 0$

Solutions:

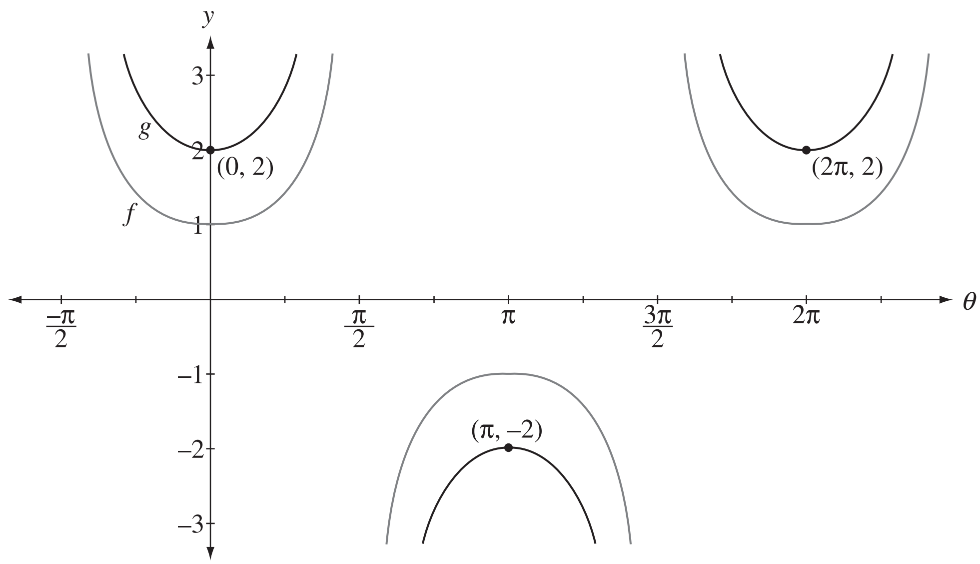
a. Amplitude is $\frac{1}{2}$, period is $\frac{2\pi}{3}$

b. $y = \frac{1}{2}\sin\left[3\left(\theta - \frac{\pi}{6}\right)\right] + 1$ or $y = -\frac{1}{2}\sin\left[3\left(\theta - \frac{\pi}{2}\right)\right] + 1$

c. $y = \frac{1}{2}\cos\left[3\left(\theta - \frac{\pi}{3}\right)\right] + 1$

SE **Numerical Response**

28. The partial graph of $f(\theta) = \sec\theta$ and the partial graph of $g(\theta) = a\sec\theta,\ a > 0,$ are shown below.



The value of a, to the nearest whole number, is __________.

Solution: 2

$a = 2$

SE **29.** Determine the domain, range, and period of $f(x) = \cot x$ and $g(x) = \cot(2x)$.

Solution:

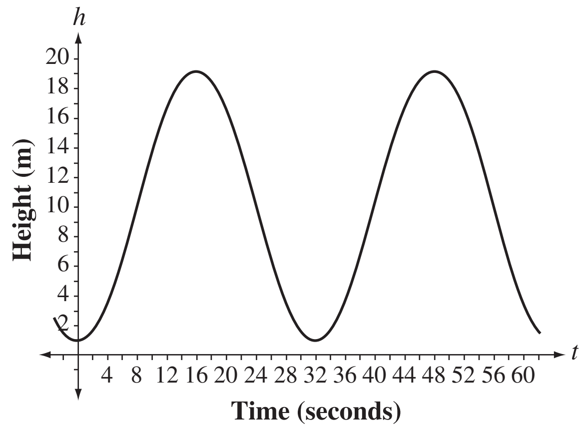
The domain of $f(x) = \cot x$ is $x \neq n\pi,\ n \in I$, and the period is π.

The domain of $g(x) = \cot(2x)$ is $x \neq \frac{n\pi}{2},\ n \in I$, and the period is $\frac{\pi}{2}$.

The range is the same for both functions: $y \in R$.

Use the following information to answer the next question.

The graph below shows the height of a particular point on a ferris wheel, h, in metres above the ground, as a function of time, t, in seconds. The maximum height of the ferris wheel is 19 m, and the minimum height is 1 m.



30. **a.** Estimate the period for 1 revolution of the ferris wheel.

b. How high is the hub or centre of the ferris wheel off the ground?

SE **c.** Write an equation for the height of the particular point on a ferris wheel, h, as a function of time, t. Use the sine function for h in terms of t.

d. Find the distance from the ground, to the nearest tenth of a metre, of the particular point on the ride at $t = 10$ s.

e. Graphically determine the first time, to the nearest tenth of a second, that the particular point on the ferris wheel is 6 m above the ground.

Solutions:

a. From the graph, it appears that the period is approximately 32 seconds.

b. Height of hub = $\frac{19+1}{2}$ = 10 m high

c. $h = 9\sin\left[\frac{2\pi}{32}(t-8)\right] + 10$

d. $h = 9\sin\left[\frac{2\pi}{32}(10-8)\right] + 10$

$h \doteq 13.4$ m

e. Let $y_1 = 9\sin\left[\frac{2\pi}{32}(t-8)\right] + 10$

and $y_2 = 6$, and use **INTERSECT** function

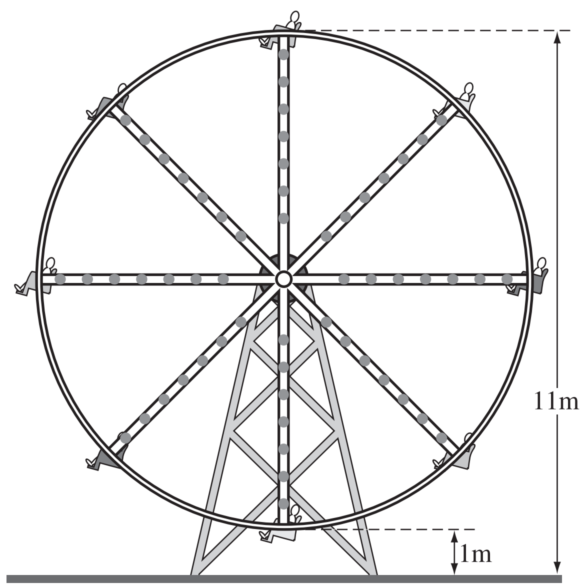
$\therefore\ t \doteq 5.7$ s.

Use the following information to answer the next question.

The distance above the ground of a passenger on a circular ferris wheel is given by the equation

$$h(t) = 5\sin\left[\frac{\pi}{12}(t - 6)\right] + 6$$

where h is the distance above the ground, in metres, and t is the time, in seconds, after the passenger passes the lowest point of the ride for the first time.



31. The distance of the passenger above the ground 10 s after passing the lowest point of the ride, to the nearest tenth of a metre, is

A. 4.6 m
B. 6.1 m
C. 8.5 m
* **D.** 10.3 m

Solution:

With calculator set in radian mode, calculate $h(10) = 5\sin\left[\frac{\pi}{12}(10-6)\right] + 6 \doteq 10.3$ m.

Conic Sections

General Outcome

Classify conic sections, using their shapes and equations.

General Notes:

- The transformations of relations from specific outcome 1.5 (Describe and perform single transformations and combinations of transformations on functions and relations) will be assessed as part of the conics unit on the diploma examination. The terminology relating to this outcome should be common to the conics unit and transformations unit.
- Students will be required to identify the conic formed by the intersection of a plane with a right circular cone.
- Students will also be required to identify the type of conic formed by a given equation in general or standard form. The intent is that students look at the graph and analyze the transformations from the standard form of the equation.
- Students must be able to recognize that degenerate conics are produced when the cutting plane intersects the vertex of a double-napped cone. Equations are not required.
- Students must be able to recognize that degenerate conics are produced when the cutting plane intersects a cylindrical region parallel to the generator. Equations are not required.
- The use of eccentricity and the derivation of the equations of conics from locus definitions are not required. Rather, the emphasis should be on defining conics in terms of their basic shapes and relating these basic shapes to the parameter values in the standard form equation and the general form equation.
- The use of the terms major, minor, transverse, and conjugate are not required.
- The use of the graphing calculator and the standard conics program or a graphing utility on a computer may be used as a teaching tool; however, calculator memories must be cleared for the diploma examination and students cannot rely on conic programs to determine the type of conic.

Specific Outcomes

Specific Outcome 4.1

Classify conic sections according to shape. [C, R, V]

4.1 Notes:

- Students must consider the conic sections and degenerate conics formed by a plane slicing a double-napped cone or a cylinder.
- Given a specific measure for the vertex angle, students should be able to determine the shape of the conic formed if the cutting plane intersects the axis of a cone (central axis) at a given angle.

(See examples 1 and 2)

Specific Outcome 4.2

Classify conic sections according to a given equation in general or standard (completed square) form (vertical or horizontal axis of symmetry only). [CN, T, V]

4.2 Notes:

- General form equation for a quadratic relation is $Ax^2 + Cy^2 + Dx + Ey + F = 0$, or $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A, C, D, E, F \in I$ and $B = 0$.
- Standard form equations for quadratic relations in completed square form are

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\left.\begin{array}{l}\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = +1 \\ \dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = \pm 1\end{array}\right\} \begin{array}{l}a \text{ represents horizontal axis} \\ b \text{ represents vertical axis}\end{array}$$

$$y - k = a(x-h)^2$$

$$x - h = a(y-k)^2$$

- Students should be able to determine the type of conic directly from the standard form equation or from the general form equation.
- The required specific terminology related to conics is centre, vertex, domain, range, intercepts, vertices of ellipse/hyperbola, and asymptotes.
- Vertices of ellipses are the end points of the longer axis.
- Students need to understand what an asymptote is but do not need to find the equations of asymptotes of a given hyperbola. When finding the equation of a hyperbola, three of the four parameters a, b, h, and k should be given.
- The location and study of focal points is not required, but students should be able to recognize the centre point of any circle, ellipse, or hyperbola, and the vertex of any parabola.

(See examples 3, 4a, and 5)

Specific Outcome 4.3

Convert a given equation of a conic section from general to standard form and vice versa. [R, T]

4.3 Notes:

- Students should be able to sketch a conic, using an equation in standard form.
- When sketching the graph of a hyperbola, asymptotes should be included.
- Students should be able to determine the equation of a conic, given a combination of key points such as vertices, intercepts, centre, and a point on the curve.
- Students should be able to determine the equation of a new conic when given a transformation.

(See examples 4b, 6, 7, 8, 9, 10, and 11)

Acceptable Standard

The student can

- describe and model the intersection of a double-napped cone and a plane for all conics, including the degenerate cases
- determine the shape of the conic formed if the cutting plane intersects the axis of the cone or central axis at a given angle, given a specific measure for the vertex angle
- determine the type of conic by examining the equation in general form and in standard form
- identify domain, range, centre, and intercepts of a conic from the equation and from the graph
- sketch the graph of a conic section when given a translation
- sketch the graph of a conic section when given the equation
- identify the conic if A and C are given in the general form of the quadratic relation
- describe the effects that changing parameters h, k, a, b in the standard form equation of the quadratic relation have on the centre and intercepts of the conic
- write the equation of a conic section when given a graph
- write the new equation of a conic section when given an equation in standard form and a translation
- convert a given equation of a non-degenerate conic section from general form to standard form and vice versa
- participate in and contribute toward the problem-solving process for problems that require the analysis of conic sections studied in Pure Mathematics 30

Standard of Excellence

The student can also

- describe and model the intersection of a cylindrical surface and a plane for all conics, including the degenerate cases
- determine a range of values for the vertex angle if the intersection is to produce a particular conic section, given a specific angle at which the cutting plane intersects the axis of the cone or central axis
- determine the type of conic by examining a combination of domain, range, centre, vertices, and intercepts
- sketch the graph of a conic section when given a combination of domain, range, centre, vertices, intercepts, and slopes of asymptotes
- sketch the graph of a conic section when given a transformation involving stretches and/or reflections
- write the equation of a conic section when given a transformation
- write the equation of a conic section when given key data such as
  - vertices
  - centre
  - intercepts
  - point on the curve
- complete the solution for problems that require the analysis of conic sections studied in Pure Mathematics 30

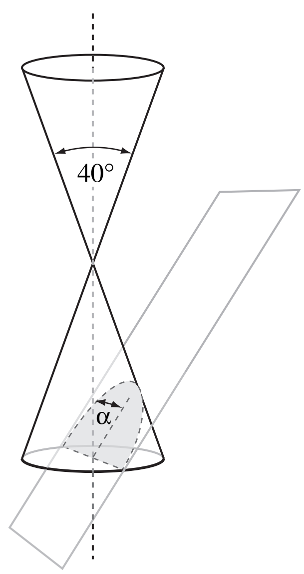
Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *standard of excellence*.

1. A right circular cone is intersected by a plane that is parallel to a generator of the cone. The shape of the conic section formed by the intersection of the plane and cone is

A. a hyperbola
* **B.** a parabola
C. an ellipse
D. a circle

SE **2.** The vertex angle of a double-napped cone is 40°. The angle α is the angle measured from the axis of the cone (central axis) to the cutting plane, as shown in the diagram. For what range of values of α must the cutting plane intersect the conical surface in order to generate a hyperbola?



Solution:

The angle between the cutting plane and the axis of the cone must be from 0° up to, but not including, 20°.

3. For the equation $x^2 - 4y^2 - 4x = 0$, state the type of conic and describe the vertices, centre, intercepts, domain, and range of the corresponding graph. Justify your answers.

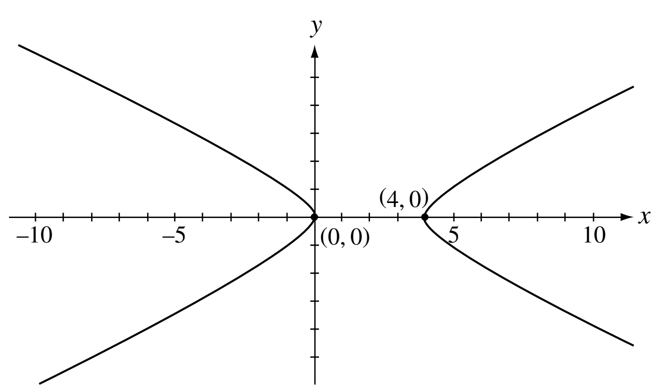
Solution:

Since $AC < 0$, the graph is a hyperbola. Letting $x = 0$ results in the single y-intercept $y = 0$. Letting $y = 0$ results in the two x-intercepts $x = 0$ and $x = 4$. Because of the location of the intercepts, we know that the hyperbola opens left and right, has vertices $(4, 0)$ and $(0, 0)$, and is centred at $(2, 0)$. Thus, the domain is $x \le 0$ or $x \ge 4$. The range is the set of real numbers.

Alternate Solution:

The standard equation form is $\frac{(x-2)^2}{4} - y^2 = 1$. This equation indicates that the graph is a hyperbola because the equation represents a difference of squares. The graph is a translated form of the graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since $a^2 = 4$ and $a = 2$, the graph of $x^2 - y^2 = 1$ is first stretched horizontally about the y-axis by a factor of two so that the new vertices are at $(\pm 2, 0)$. The graph is then translated two units to the right so that the vertices are now at (0, 0) and (4, 0) and the centre is at (2, 0). Thus, the domain is $x \le 0$ or $x \ge 4$, and the range is the set of real numbers.

Alternate Solution (Graphing):



The characteristics of this graph are such that the vertices are at (0, 0) and (4, 0) and the centre is at (2, 0). The domain is $x \le 0$ or $x \ge 4$, and the range is the set of real numbers.

SE **4.** A particular hyperbola in standard form of $\frac{(x-h)^2}{9} - \frac{(y-k)^2}{b^2} = -1$ has a centre point of $(2, -1)$ and goes through the point $(2, -2)$.

a. What is the standard form equation of the hyperbola?
b. Sketch the graph, identifying key data.

Solutions:

a.
$$\frac{(x-h)^2}{9} - \frac{(y-k)^2}{b^2} = -1$$

Substitute point (2, –2) and the centre point (2, –1)

$$\frac{(2-2)^2}{9} - \frac{(-2+1)^2}{b^2} = -1$$

$$-\frac{(-2+1)^2}{b^2} = -1$$

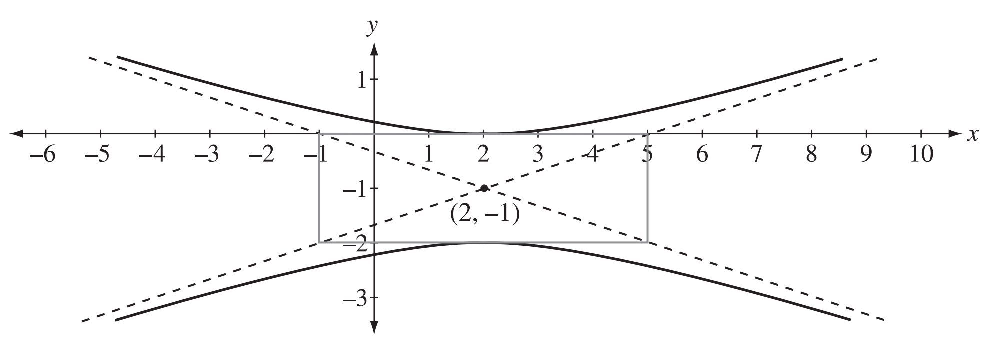
$$(-1)^2 = b^2$$

$$1 = b^2$$

$$1 = b \text{ since } b > 0$$

$$\frac{(x-2)^2}{9} - \frac{(y+1)^2}{1} = -1$$

b. Sketch the graph identifying key data. Two methods – rectangles with diagonals
– slopes of asymptotes



SE **5.** A hyperbola has its centre at $(0, 0)$, and one of its vertices at the point $(0, 3)$. If $b = 3a$ where a and $b > 0$, then the equation of the hyperbola is

A. $\frac{y^2}{81} - x^2 = 1$

* **B.** $\frac{y^2}{9} - x^2 = 1$

C. $\frac{y^2}{81} - 9x^2 = 1$

D. $\frac{y^2}{9} - 9x^2 = 1$

Solution:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$$

$$\frac{x^2}{a^2} - \frac{y^2}{9a^2} = -1$$

$$\frac{0^2}{a^2} - \frac{3^2}{9a^2} = -1$$

$$-9 = -9a^2$$

$$a^2 = 1$$

$$a = 1, \text{ since } a > 0$$

$$b = 3a$$

$$b = 3$$

Therefore, $\frac{x^2}{1} - \frac{y^2}{9} = -1$

6. If the equation of a particular ellipse is $3x^2 + y^2 - 12x + 10y = 0$, then the centre of this ellipse is point

A. (6, 5)
* **B.** (2, –5)
C. (–6, 5)
D. (–2, –5)

Solution:

$$3(x^2 - 4x + 4) - 12 + (y^2 + 10y + 25) - 25 = 0$$
$$3(x-2)^2 + (y+5)^2 = 37$$
$$\frac{3(x-2)^2}{37} + \frac{(y+5)^2}{37} = 1$$

The centre is (2, –5).

7. When a circle represented by the equation $x^2 + 10x + y^2 - 8y = 11$ is translated 7 units to the right, the new circle has its centre at

A. (12, –4)
B. (–12, 4)
C. (–2, –4)
* **D.** (2, 4)

Solution:

$$(x^2 + 10x + 25) + (y^2 - 8y + 16) = 11 + 25 + 16$$
$$(x+5)^2 + (y-4)^2 = 52$$

centre: (–5, 4)

When the graph is moved 7 units to the right, the centre becomes (2, 4).

SE **8.** Determine the general form equation of an ellipse that is formed if $x^2 + 5y^2 - 25 = 0$ is translated 3 units down.

Solution:

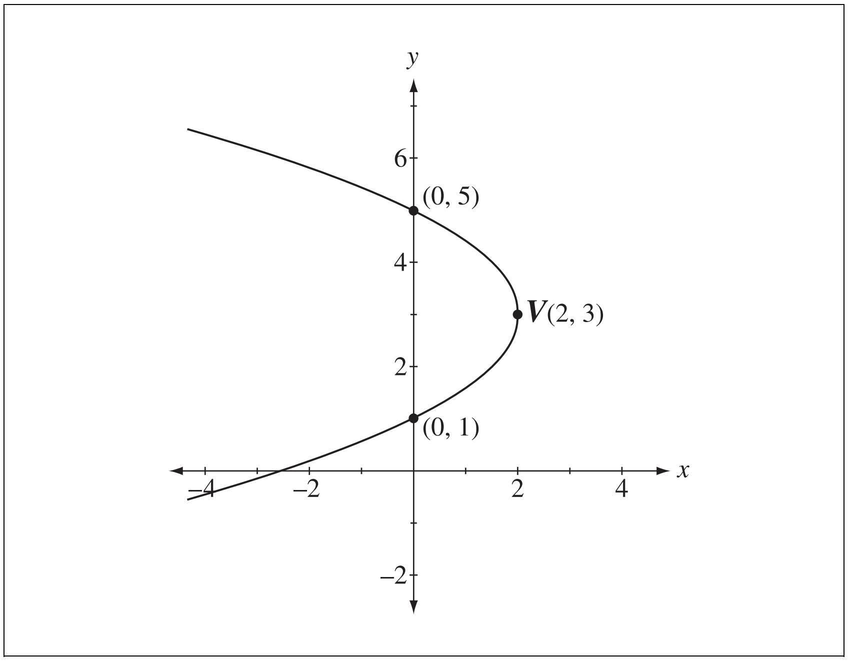
$$
\begin{aligned}
x^2 + 5y^2 &= 25 \\
\frac{x^2}{25} + \frac{y^2}{5} &= 1 && \text{(standard form)} \\
\frac{x^2}{25} + \frac{(y+3)^2}{5} &= 1 && \text{(translated down 3 units)} \\
x^2 + 5(y+3)^2 &= 25 \\
x^2 + 5(y^2 + 6y + 9) &= 25 \\
x^2 + 5y^2 + 30y + 45 &= 25 \\
x^2 + 5y^2 + 30y + 20 &= 0 && \text{(general form)}
\end{aligned}
$$

or

Replace y by $y + 3$

$$
\begin{aligned}
x^2 + 5y^2 - 25 &= 0 && \text{original} \\
x^2 + 5(y+3)^2 - 25 &= 0 && \text{(translated 3 units down)} \\
x^2 + 5(y^2 + 6y + 9) - 25 &= 0 \\
x^2 + 5y^2 + 30y + 45 - 25 &= 0 \\
x^2 + 5y^2 + 30y + 20 &= 0
\end{aligned}
$$

Use the following conic to answer the next question.



9. Find the general form equation of the conic above.

Solution:

Standard form equation is

$$x - h = a(y - k)^2$$
$$x - 2 = a(y - 3)^2$$

Substitute one of the given points, (0, 1), to find the a value.

$$0 - 2 = a(1 - 3)^2$$
$$-2 = a(-2)^2$$
$$-\frac{1}{2} = a$$

The standard form equation is $x - 2 = -\frac{1}{2}(y - 3)^2$

The general form equation is $y^2 + 2x - 6y + 5 = 0$

10. A conic has the domain of $0 \le x \le 8$ and the range of $4 \le y \le 8$.

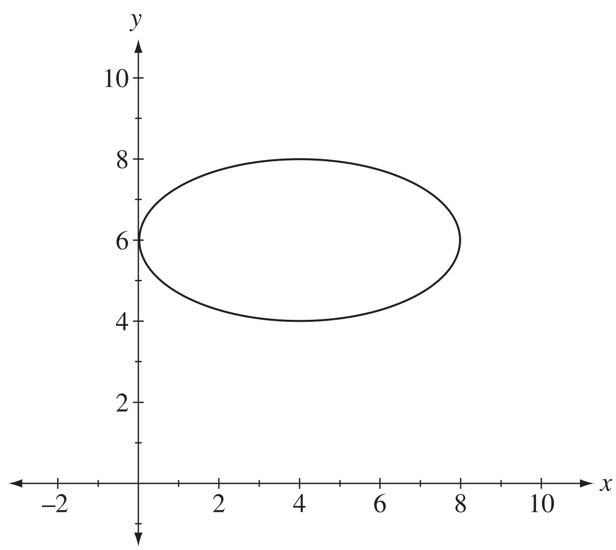
a. Sketch the graph of the conic.

b. Determine the standard form equation of this conic.

SE **c.** If the conic is stretched vertically by a factor of 2 about the line $y = 6$, then what is the standard form equation of the new transformed conic?

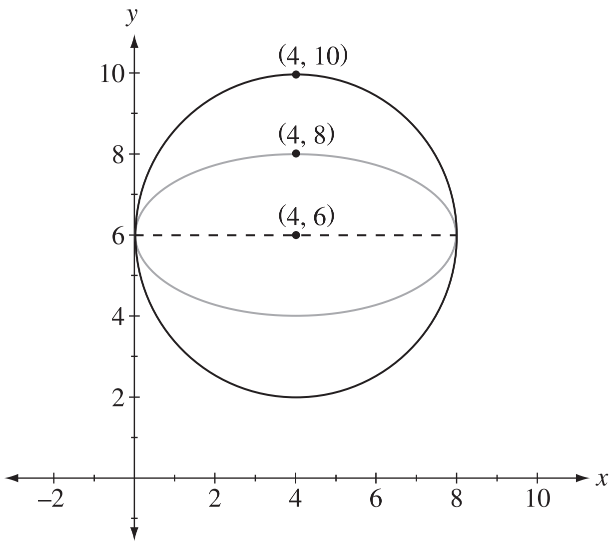
Solutions:

a.



b. The equation in standard form is $\dfrac{(x-4)^2}{16} + \dfrac{(y-6)^2}{4} = 1.$

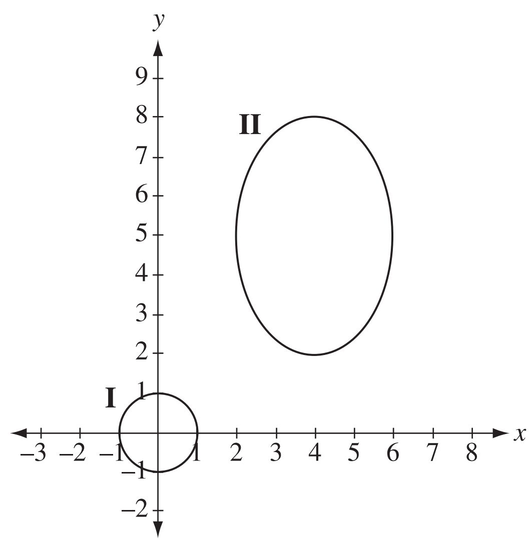
c.



The centre of the transformed conic stays the same and the endpoints of the horizontal axis stay the same, because these points are all located on the stretch line. The endpoints of the vertical axis are 2 units from $y = 6$, so after applying the vertical stretch factor of 2, the endpoint will now be 4 units from $y = 6$. Therefore, the transformed conic is a circle with centre (4, 6) and a radius of 4 units.

The standard form equation is $(x - 4)^2 + (y - 6)^2 = 16$.

Use the following graphs to answer the next question.



11. Given the graph of a circle with centre at (0, 0) and radius 1, as shown in graph I above, describe the series of transformations required to transform graph I into graph II, which is an ellipse with centre at (4, 5), a length of 6 units measured parallel to the y-axis, and a width of 4 units measured parallel to the x-axis.

Solution:

Graph I has to be stretched vertically about the x-axis by a factor of 3 and stretched horizontally about the y-axis by a factor of 2. The graph is then translated 4 units to the right and 5 units up.

or

Graph I is translated 4 units to the right and 5 units up. The graph is then stretched vertically about the line $y = 5$ by a factor of 3 and stretched horizontally about the line $x = 4$ by a factor of 2.

Permutations and Combinations

General Outcome

Solve problems based on the counting of sets, using techniques such as the fundamental counting principle, permutations, and combinations.

Model the probability of a compound event, and solve problems based on the combining of simpler probabilities.

General Notes:

- Particular attention must be paid to the use of the words *and*, *or*, *at least*, *at most*, and *no more than*.
- Certain outcomes may need to have supplemental materials.
- The context of problems will be described so that if students are not familiar with the scenario, they will not be disadvantaged.
- It is expected that students are able to manipulate the ${}_nP_r$ and ${}_nC_r$ formulas algebraically.

Specific Outcomes

Specific Outcome 5.1

Use the fundamental counting principle to determine the number of different ways to perform multistep operations. [PS, R]

(See example 1)

Specific Outcome 5.2

Determine the number of linear permutations of n objects taken r at a time, and use this to solve problems. [PS, R, V]

5.2 Notes:

- Permutation problems should involve repetition (like elements) and restrictions. The approved resources may need to be supplemented.
- Circular and ring permutations are not part of this outcome.
- ${}_nP_r$ assumes no like elements and no restrictions

(See examples 2, 3, and 4)

Specific Outcome 5.3

Determine the number of combinations of n distinguishable objects taken r at a time, and use this to solve problems. [PS, R, V]

(See examples 5, 6, 7, 8, 9, 10, and 11)

Specific Outcome 5.4

Determine the number of pathways in a given simple pathway problem. [CN, PS, R, V]

5.4 Notes:

- Students are to find solutions through visualization, the use of the fundamental counting principle, permutations, combinations, or the use of Pascal's triangle.
- Approved resources have very few questions involving pathways and may need to be supplemented.

(See example 12)

Specific Outcome 5.5

Determine the number of pathways in a given compound pathway problem. [CN, PS, R, V]

5.5 Notes:

- More complicated pathways should be solved by using permutations or combinations in some cases (three-dimensional pathways) and Pascal's triangle in other cases (overlapping pathways, grids with holes, and partial grids).
- Approved resources have very few questions involving compound pathways and may need to be supplemented.
- Pathways do not always have to be rectangular in shape; some edges or paths may be curved.

(See examples 13, 14, 15, and 16)

Specific Outcome 5.6

Solve problems using the binomial theorem, where the exponent n belongs to the set of natural numbers. [CN, E, PS, V]

5.6 Notes:

- Students must explore the relationship between Pascal's triangle and the numerical coefficients of the terms in the expansion of a binomial.
- Students must have a good working knowledge of all exponent laws.
- There are approved calculators that will perform the algebraic processes for many of the examples, such as 17 and 18; therefore, this outcome will be assessed in a different manner on the diploma examination.

(See examples 17, 18, 19, and 20)

Specific Outcome 5.7

Solve probability problems using either permutations and combinations or the fundamental counting principle. [E, PS, R]

5.7 Notes:

- Probability problems involving **both** permutations and combinations are not an intended outcome.
- All probability questions must be able to be solved using the fundamental counting principle, permutations, or combinations.
- It is not necessary for students to know all methods; however, students who do know all methods may have an advantage.
- Students may need to review basic concepts and terminology of probability related to independent events, dependent events, and complementary events.
- Students are not expected to solve problems involving events that are not mutually exclusive.
- Probability should be expressed as a decimal or a fractional value between 0 and 1 unless otherwise specified.
- Conditional probability is not an expected outcome for this course.

(See examples 21, 22, and 23)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• apply the fundamental counting principle to various straightforward multistep problems with few constraints	• recognize and address constraints and ambiguities within multistep problems
• understand and use factorial notation	
• calculate the permutations of n things taken r at a time using ${}_nP_r$	
• obtain solutions to problems involving a single case or constraint	• obtain complete solutions to problems involving two or more cases or constraints
• calculate the number of combinations of n things taken r at a time using ${}_nC_r$	
• solve for n in equations involving ${}_nC_r$ or ${}_nP_r$, given r	• solve for n in equations involving ${}_nC_r$ and ${}_nP_r$, given r • solve problems involving both permutations and combinations
• solve simple pathway problems	• solve three-dimensional pathway problems
• solve compound two-dimensional pathway problems with a given diagram	• solve two-dimensional pathway problems without a given diagram. They can also generalize the results for an $x \times y$ grid
• expand $(x + y)^n$, where $n \in N$	• expand $(x + y)^n$, where x and y can be monomials involving coefficients and powers
• determine specified terms of an expansion with linear terms within the binomial	• determine specified terms of an expansion with non-linear terms within the binomial • determine the coefficient of a specified term of an expansion with non-linear terms within the binomial
• determine probability by using the fundamental counting principle where order is important	• determine probability by using the fundamental counting principle where order is not important
• determine the probability of one event in which permutations or combinations are involved	• determine the probabilities for two or more events in which permutations or combinations are involved
• participate in and contribute toward the problem-solving process for problems involving permutations or combinations studied in Pure Mathematics 30	• complete all solutions in the problem-solving process for problems involving permutations or combinations studied in Pure Mathematics 30
• participate in and contribute toward the problem-solving process for probability problems studied in Pure Mathematics 30	• complete the solution to probability problems studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *standard of excellence*.

SE **1.** If all of the letters in the word **DIPLOMA** are used, then the number of different 7-letter arrangements that can be made beginning with 3 vowels is

A. 24
* **B.** 144
C. 720
D. 5 040

Solution:

$$\underset{(v)}{3}\bullet\underset{(v)}{2}\bullet\underset{(v)}{1}\bullet\underset{(c)}{4}\bullet\underset{(c)}{3}\bullet\underset{(c)}{2}\bullet\underset{(c)}{1}$$

$$3!4! = 144$$

2. A car manager wants to line up 10 cars that are identical except for colour. There are 3 red cars, 2 blue cars, and 5 green cars. Determine the number of possible arrangements of the 10 cars if they are lined up in a row along one side of a parking lot, and a blue car is parked on each end of the row.

Solution:

$$\frac{2 \times 8! \times 1}{2!3!5!} = 56$$

3. A 6-player volleyball team stands in a straight line for a picture. If 2 particular players, Joan and Emily, must be together, then how many different arrangements can be made for the picture?

Solution:

$$5!2! = 240$$

4. A teacher tells his students that on a multiple-choice test with 12 questions, 2 answers are **A**, 3 are **B**, 3 are **C**, and 4 are **D**. How many different answer keys are possible?

Solution:

$$\frac{12!}{2!3!3!4!} = 277\ 200$$

Use the following information to answer the next question.

At a particular hotel, the following items are available for the continental breakfast.

Beverage	**Pastry**	**Fruit**
Coffee	Muffin	Apple
Tea	Toast	Orange
Juice	Doughnut	Grapefruit
		Banana

5. If the continental breakfast consists of 1 beverage, 1 pastry, and 2 different types of fruit, then the number of possible breakfasts that can be ordered is

* **A.** ${}_3C_1 \times {}_3C_1 \times {}_4C_2$
B. ${}_3P_1 \times {}_3P_1 \times {}_4P_2$
C. ${}_{10}C_4$
D. ${}_{10}P_4$

SE **6.** Solve ${}_nC_2 = \frac{{}_nP_3}{3!}$, and verify your solution.

Solution:

$$\frac{n(n-1)(n-2)!}{(n-2)!\,2!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!\,3!}$$

$$n(n-1)3! = n(n-1)(n-2)2!$$

$$3! = (n-2)2!,\ n \neq 0, 1$$

$$6 = 2n - 4$$

$$10 = 2n$$

$$n = 5$$

LHS	RHS
${}_5C_2 = 10$	$\frac{{}_5P_3}{3!} = \frac{60}{6} = 10$

7. A school committee consists of 1 vice-principal, 2 teachers, and 3 students. The number of different committees that can be selected from 2 vice-principals, 5 teachers, and 9 students is

A. 20 160
B. 8 008
* **C.** 1 680
D. 90

Solution:

$$\underset{\text{(VP)}}{{}_2C_1} \times \underset{\text{(Teachers)}}{{}_5C_2} \times \underset{\text{(Students)}}{{}_9C_3} = 2 \times 10 \times 84 = 1\ 680$$

8. In a basketball league there are 6 teams. In league play, each team must play every other team twice. Determine the number of games that must be scheduled.

Solution:

$2 \times {}_6C_2 = 30$ games

9. The vertices of an octagon are marked on a circle. Determine the number of triangles that can be formed using any 3 of the vertices.

Solution:

$_8C_3 = 56$

SE **10.** In a group of 9 people, there are 4 females and 5 males. Determine the number of 4-member committees consisting of at least 1 female that can be formed.

Solution:

Method 1
Total number of committees possible – number of committees with no females.
$_9C_4 - {_5C_4} = 126 - 5 = 121$

Method 2
1 female and 3 males
or 2 females and 2 males
or 3 females and 1 male
or 4 females and 0 males

$= ({_4C_1} \times {_5C_3}) + ({_4C_2} \times {_5C_2}) + ({_4C_3} \times {_5C_1}) + ({_4C_4} \times {_5C_0})$
$= 40 + 60 + 20 + 1$
$= 121$

SE **11.** How many different arrangements of the letters **TOFIELD** can be made using exactly 2 vowels and exactly 2 consonants?

Solution:

Choose the letters and then arrange them.

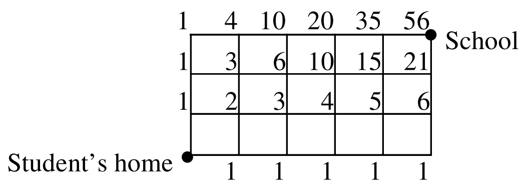
$\underset{\text{(V)}}{{_3C_2}} \times \underset{\text{(C)}}{{_4C_2}} \times \underset{\text{(arrangements)}}{4!} = 3 \times 6 \times 24 \text{ or } 432$

SE **Numerical Response**

12. In a particular town, all of the streets run north–south or east–west. A student must travel 5 blocks east and 3 blocks north to arrive at school. The number of different routes, 8 blocks in length, that the student can take to get to the school is __________.

Solution:

Method 1
Draw a diagram similar to the one below. Count the number of paths to each corner in the grid, like Pascal's triangle. Since the diagram was not provided, this is considered an *excellence* level question.

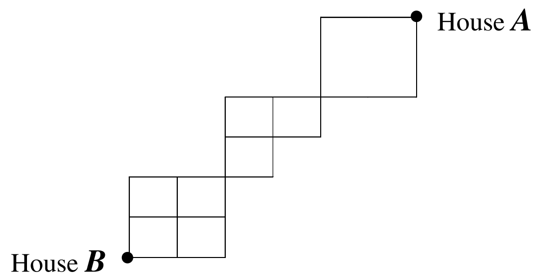


Method 2
Label the 5 blocks east as EEEEE and the 3 blocks north as NNN. There are 8 blocks in total with 5 repetitions of E and 3 repetitions of N.

$$\frac{8!}{5!3!} = 56$$

Use the following information to answer the next question.

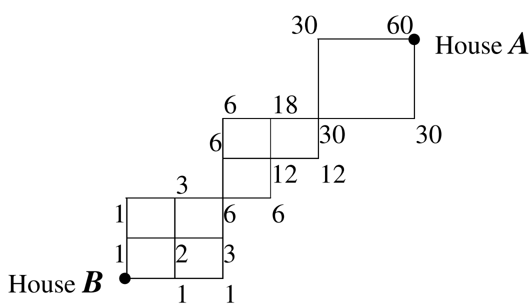
A newspaper carrier who delivers papers on his bike can travel only on the trails represented in the diagram below.



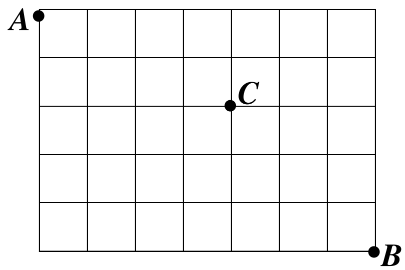
13. The number of different trails that the newspaper carrier can take to get from house B to house A without backtracking is

A. 13
B. 32
* **C.** 60
D. 72

Solution:



14. Given the diagram below, determine the number of pathways starting from *A* and moving to *B* along the gridlines if a pathway must pass through *C* and must always move closer to *B*.



Solution:

Method 1

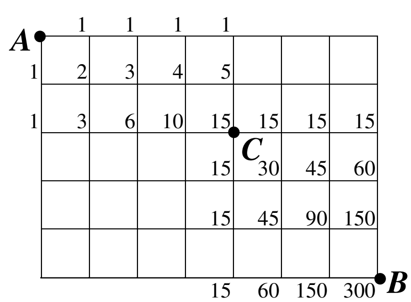
Using combinations

$$\frac{6!}{4!2!}\times\frac{6!}{3!3!}=15\times20=300$$

(*A* to *C*)(*C* to *B*)

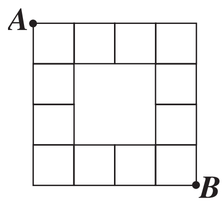
Method 2

Counting using properties of Pascal's triangle.

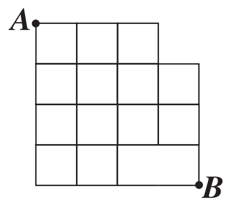


15. For each of the diagrams below, determine the number of pathways starting from *A* and moving to *B* along the gridlines given that a pathway must always move closer to *B*.

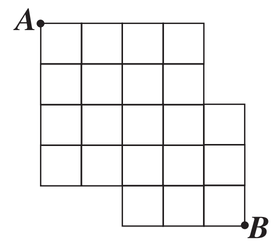
I



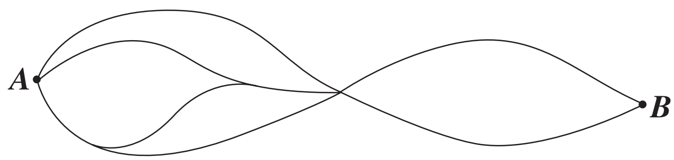
II



III

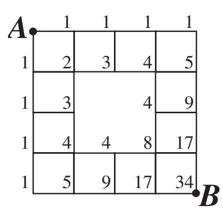


IV

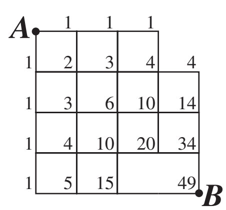


Solution:

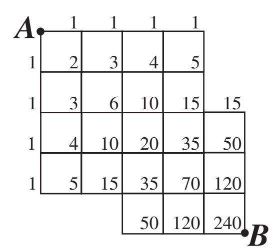
I



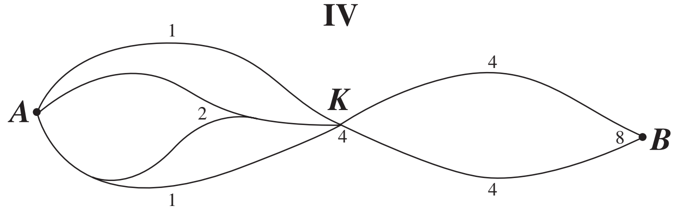
II



III

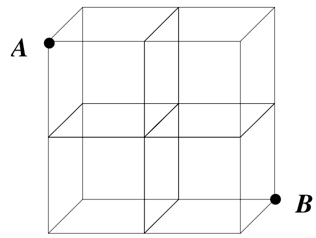


IV

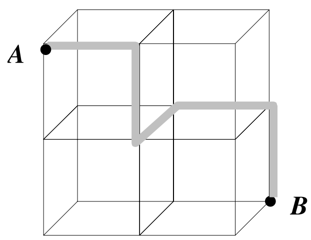


Since three paths lead into point *K*, three values must be added.
Since two paths lead into point *B*, two values must be added.

SE **16.** Determine the number of possible paths from point A to point B in the following diagram. Travel may occur only along the edges of the cubes and the path must always move closer to B.



Solution:



One possible path is shown. Any path from A to B must have 5 edges and must consist of 2 edges to the right (R), 2 edges downward (D), and 1 edge to the back (B).

The number of arrangements of these letters is
$\frac{5!}{2!2!1!} = \frac{120}{4} = 30$.

Therefore, 30 paths are possible.

17. Determine the coefficient of the term containing Ax^2y^5 in the expansion of $(2x + y)^7$.

Solution:

${}_7C_k(2x)^{7-k}(y)^k = Ax^2y^5$
since $x^{7-k} = x^2$ **or** $y^k = y^5$
Therefore, $k = 5$

$$\begin{aligned} {}_7C_5(2x)^2(y)^5 &= Ax^2y^5 \\ (21)(4x^2)(y^5) &= Ax^2y^5 \\ 84x^2y^5 &= Ax^2y^5 \\ A &= 84 \end{aligned}$$

18. Expand $(2a - 3)^4$ by using the binomial theorem.

Solution:

$$\begin{aligned} t_{k+1} &= {}_4C_k(2a)^{4-k}(-3)^k \\ &= {}_4C_0(2a)^4 + {}_4C_1(2a)^3(-3) + {}_4C_2(2a)^2(-3)^2 + {}_4C_3(2a)(-3)^3 + {}_4C_4(-3)^4 \\ &= 16a^4 - 96a^3 + 216a^2 - 216a + 81 \end{aligned}$$

SE **19.** Determine the general term and the constant term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^6$.

Solution:

General Term

$$\begin{aligned} t_{k+1} &= {}_nC_k p^{n-k} q^k \\ &= {}_6C_k (2x^2)^{6-k}\left(-\frac{1}{x}\right)^k \\ &= {}_6C_k 2^{6-k} x^{12-2k} \frac{(-1)^k}{x^k} \\ &= {}_6C_k 2^{6-k} (-1)^k x^{12-3k} \end{aligned}$$

Constant Term (A constant implies x^0)

$$\begin{aligned} \therefore\ 12 - 3k &= 0 \\ 12 &= 3k \\ 4 &= k \end{aligned}$$

$$\begin{aligned} \therefore\ {}_6C_4(2x^2)^2\left(-\frac{1}{x}\right)^4 &= 15(4x^4)\left(\frac{1}{x^4}\right) \\ &= 60 \end{aligned}$$

SE **Numerical Response**

20. A term of the binomial expansion $(ax + y)^8$, where $a > 0$, is $112x^2y^6$. The value of a, correct to the nearest whole number, is __________.

Solution: 2

$$\begin{aligned} {}_8C_6(ax)^2(y^6) &= 112x^2y^6 \\ 28a^2x^2y^6 &= 112x^2y^6 \\ 28a^2 &= 112 \\ a^2 &= 4 \\ a &= 2 \end{aligned}$$

Use the following information to answer the next question.

At a family reunion, door prizes are to be given out. At one table in the community hall, 6 children, 3 teenagers, 4 adults, and 5 seniors are seated. The 3 winning tickets are held by 3 different people at this table.

21. The probability that the 3 winning tickets are held by 3 people in the same age group, correct to the nearest thousandth, is

A. 0.037
* **B.** 0.043
C. 0.222
D. 0.980

Solution:

The 3 winning tickets could be held by the children, the teenagers, the adults, or the seniors.

If the door prizes are all different, then $\dfrac{{}_6P_3 + {}_3P_3 + {}_4P_3 + {}_5P_3}{{}_{18}P_3} \doteq 0.043$.

If the door prizes are the same, then $\dfrac{{}_6C_3 + {}_3C_3 + {}_4C_3 + {}_5C_3}{{}_{18}C_3} \doteq 0.043$.

22. A bag contains 6 red marbles and 10 green marbles. From the bag, 2 marbles are drawn without replacement. Determine the probability of drawing

a. 1 red and then 1 green marble
b. 1 red and 1 green marble
c. at least 1 green marble

Solutions:

(Using Permutations and Combinations)

a. 1 red and then 1 green marble

$$P(\text{1 red then 1 green}) = \frac{{}_6P_1 \times {}_{10}P_1}{{}_{16}P_2} = \frac{1}{4}$$

b. 1 red and 1 green marble

$$P(\text{1 red and 1 green}) = \frac{{}_6C_1 \times {}_{10}C_1}{{}_{16}C_2} = \frac{1}{2}$$

c. at least 1 green marble

$$P(\text{at least 1 green}) = \frac{{}_6C_1 \times {}_{10}C_1 + {}_{10}C_2 \times {}_6C_0}{{}_{16}C_2} = \frac{7}{8}$$

or

$$\begin{aligned} P(\text{at least 1 green}) &= 1 - P(\text{no green}) \\ &= 1 - \frac{{}_6C_2}{{}_{16}C_2} \\ &= 1 - \frac{15}{120} = \frac{7}{8} \end{aligned}$$

or
(Using Fundamental Counting Principle)

a. 1 red and then 1 green marble

$$P(\text{1 red then 1 green}) = \frac{6}{16} \times \frac{10}{15} = \frac{1}{4}$$

b. 1 red and 1 green marble

$$P(\text{1 red and 1 green}) = \frac{6}{16} \times \frac{10}{15} + \frac{10}{16} \times \frac{6}{15} \quad \text{or} \quad \left(\frac{6}{16} \times \frac{10}{15}\right) \times 2! = \frac{1}{2}$$

c. at least 1 green marble

$$P(\text{at least 1 green}) = 1 - P(\text{no green}) = 1 - \frac{6}{16} \times \frac{5}{15} = \frac{7}{8}$$

Use the following information to answer the next question.

Peter places the 5 equal-sized tiles shown below in a cloth bag.

P **E** **T** **E** **R**

SE **Numerical Response**

23. The probability, correct to the nearest hundredth, that Peter selects the 5 tiles, one at a time, in order such that they spell **PETER** is __________.

Solution: 0.02

Since the 2 Es are indistinguishable, repetition is involved.

$$P(\text{spell Peter}) = \frac{\frac{2}{2!}}{\frac{5!}{2!}} \doteq 0.02$$

or

$$\frac{1}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{60} \doteq 0.02$$

Statistics

General Outcome

Use normal and binomial probability distributions to solve problems involving uncertainty.

General Notes:

- The students should be using their calculators to enter data and to calculate statistical solutions.
- Review measures of central tendency, routine probability questions, and the difference between a sample and a population.
- All data are considered to be exact and discrete unless otherwise noted.
- Data distribution may include histograms, tabled data, frequency polygons, and raw data.
- Teachers may want to provide their students with calculator commands to find mean, standard deviation, binomialpdf, etc.
- Teachers may want to refer students using TI-83 or TI-83 Plus calculators to the Utilities section (p. 345) of *Applied Mathematics 12* for detailed instructions on calculator commands for some of the statistics outcomes.

Specific Outcomes

Specific Outcome 6.1

Find the population standard deviation of a data set, using technology. [CN, E, T, V]

6.1 Note:

- Sample standard deviation will not be used. Population mean is μ and standard deviation is represented by σ. Some resources use $\bar{x}$ for the population mean.

(See examples 1, 2, and 3)

Specific Outcome 6.2

Solve probability problems, using the binomial distribution. [PS, R, T]

6.2 Notes:

- A binomial experiment is one that consists of n identical trials where each trial results in one of two outcomes. The probability of success of any trial remains constant throughout the experiment. The sum of the probabilities of these outcomes is 1 and the trials are independent.
- A binomial distribution is the sequence of all possible probabilities resulting from a binomial experiment.
- Probability should be expressed as a decimal or a fractional value between 0 and 1 unless otherwise specified.

(See examples 4, 5, 6, and 7)

Specific Outcome 6.3

Use z-scores to solve problems related to the normal distribution. [PS, R, T, V]

6.3 Notes:

- The z-score tables are organized differently in the two recommended resources.
- Teachers and students need to be aware that the z-scores and areas under the curve from z-score tables may vary slightly from those values given by the calculator. When writing diploma examinations, students may use either technology **or** z-score tables.
- Some graphing calculators on the approved list do not have the capability of calculating and drawing z-score probability curves; therefore, students with these models of calculators must use tables.

(See examples 8, 9, and 10)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• determine population standard deviation and mean by using technology	
• answer questions related to given data and compare with the normal distribution	
• rank standard deviations without any calculations, given various data distributions	• interpret how changing data can affect the standard deviation and/or mean
• find areas under the normal curve when given μ and σ	
• calculate z-scores by using the formula	
• calculate the missing value when given the z-score and either μ or σ	
• calculate the area between any two z-scores	• given a specified area under the curve and one z-score, find the other z-score
• use z-scores to compare two sets of data and draw conclusions	
• determine probability using the binomial distribution for a single trial and for exact number of successes on multiple trials	• determine probabilities using the binomial distribution for multiple trials and for an exact (at least, at most) number of successes on multiple trials
• determine when to use binomial distribution, normal distribution, or probability model to solve problems	
• participate in and contribute toward the problem-solving process for problems that require the analysis of statistics studied in Pure Mathematics 30	• complete the solution to problems that require the analysis of statistics studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *standard of excellence*.

Use the following information to answer the next question.

On the table below, a school counsellor has recorded the percentage of each school day, for 8 days, that he spent talking with students.

Day	1	2	3	4	5	6	7	8
%	90	80	75	68	88	92	85	82

Numerical Response

1. The value of the standard deviation for the percentages, correct to the nearest hundredth, is __________.

Solution: 7.55

The **population** standard deviation of 90, 80, 75, 68, 88, 92, 85, 82 is 7.55.

2. A set of 100 numbers has a standard deviation of 10. To each of the numbers, 5 is added.

a. What effect does this addition have on the standard deviation?

SE **b.** How could the standard deviation be increased to 12?

Solutions:

a. No effect on standard deviation.

b. Change all data values to *z*-scores. Then, using these *z*-scores, the same mean, and a standard deviation of 12, calculate the new values.

3. If a set of data has a standard deviation of 0, then

A. the mean of the data must be 0
* **B.** all of the data values are the same
C. the data values collected have a sum of 0
D. the z-score of the mean of the data is equal to 1

Solution:

When a set of data has a standard deviation of 0, the data has no spread from the mean.

4. A certain soccer player has scored on 82% of his penalty kicks throughout his career. Given this information, the probability that he will score on exactly 4 of his next 5 penalty kicks, correct to the nearest hundredth, is

A. 0.80
B. 0.66
* **C.** 0.41
D. 0.08

Solution:

$$\begin{aligned} P(4) &= {}_5C_4(0.82)^4(0.18)^1 \\ &\doteq 0.4069 \\ &\doteq 0.41 \end{aligned}$$

or

$$\begin{aligned} P(4) &= \text{binompdf}\,(5, 0.82, 4) \\ &\doteq 0.4069 \\ &\doteq 0.41 \end{aligned}$$

SE **Numerical Response**

5. In a particular school, 80% of the students are bussed to school. If a statistician randomly sampled 50 students from the school, then the probability that at least 46 of these students are bussed to school, correct to the nearest hundredth, would be __________.

Solution: 0.02

$$1 - \text{binomcdf}(50,0.80,45) \doteq 0.01849\ldots$$
$$\doteq 0.02$$

or

$${}_{50}C_{46}(0.8)^{46}(0.2)^4 + {}_{50}C_{47}(0.8)^{47}(0.2)^3 + {}_{50}C_{48}(0.8)^{48}(0.2)^2 + {}_{50}C_{49}(0.8)^{49}(0.2) + {}_{50}C_{50}(0.8)^{50}(0.2)^0$$
$$\doteq 0.01849\ldots$$
$$\doteq 0.02$$

Note: Students are not expected to complete this question by using a normal approximation of the binomial distribution.

Numerical Response

6. In a group of 8 laser printers, only 6 work. If a sample of 4 printers is taken, what is the probability that exactly 3 work?

Solution:

$$\frac{{}_6C_3 \times {}_2C_1}{{}_8C_4} = \frac{40}{70} = \frac{4}{7}$$

Since the probability changes as soon as one printer is selected for the sample, binomial probability cannot be used.

Incorrect Solution:

${}_4C_3(0.75)^3(0.25)$

Numerical Response

7. A company knows that at any time in its production process, 95% of its laser printers work. What is the probability, to the nearest hundredth, that in a sample of 4 printers, exactly 3 work?

Solution:

Since the probability that a laser printer works does not change after a printer is chosen, binomial distribution can be used.

${}_4C_3(0.95)^3(0.05) \doteq 0.17$

SE **Numerical Response**

8. Jack has a test score of 75% on a test with a mean of 65%. To get into college, he needs to be in the top 10% of those who took the test. What is the largest standard deviation, to the nearest hundredth, that will allow Jack to be in the top 10%?

Solution: 7.75

P(Jack is in top 10%) implies that 90% of the area under the normal curve is to the left of his test score.

$\therefore$ Area = 0.9000

This area corresponds to a z-score of 1.28.

$$1.28 = \frac{75 - 65}{\sigma}$$

$$\sigma = \frac{10}{1.28} \doteq 7.75$$

$$\sigma = 7.75$$

SE **Numerical Response**

9. The results of a test were normally distributed with a mean of 57% and a standard deviation of 16%. The marks need to be adjusted so that the mean is 60% with a standard deviation of 12%. How can this be done?

Solution:

Step 1 Add 3% to each test result, x. Now $\mu = 60$.
Step 2 Calculate z-scores for all these new test results by using $\mu = 60$ and $\sigma = 16$.
Step 3 Use these z-scores with $\mu = 60$ and $\sigma = 12$ to produce new test results (x values).

For example, if the original mark is 65%:

Step 1 mark becomes $65 + 3 = 68\%$.

Step 2 $z = \dfrac{68 - 60}{16} = 0.5$

Step 3 $0.5 = \dfrac{x - 60}{12}$

$x = 66\%$

The new test mark is 66%.

Note: Students will notice that not much change occurs to marks near the mean and that marks farther from the mean change the most.

10. The scores for a particular examination are normally distributed with a mean of 67.4% and a standard deviation of 10.5%. What is the probability that a student who wrote the examination had a mark of 80% or less?

* **A.** 0.88
B. 0.51
C. 0.38
D. 0.21

Solution:

$x = 80\%$
$\mu = 67.4\%$
$\sigma = 10.5\%$

$$\begin{aligned} z &= \frac{80 - 67.4}{10.5} \\ &= 1.2 \end{aligned}$$

Area in table for $z = 1.2$ is 0.8849; therefore, the probability of a student having a mark of 80% or less is approximately 0.88.

or

Using a normal cumulative distribution function on a calculator, input an appropriate lower boundary (e.g., 0), an upper boundary of 80, a mean of 67.4, and a standard deviation of 10.5, to obtain a probability of 0.8849 or approximately 0.88.

11. A teacher recorded the height of each six-year-old girl in her school. She determined that their heights were normally distributed with a mean of 104 cm and a standard deviation of 3 cm. If 15% of the six-year-old girls have heights less than Samantha's, then, to the nearest centimetre, how tall is Samantha?

Solution: 101 cm

Calculator solution:

invNorm(0.15, 104, 3)
= 100.89…
$\doteq$ 101 cm

or

Using the z-score table:

The area under the curve is 15% for $z = -1.04$.

$$-1.04 = \frac{x-104}{3}$$

$$x = 100.88$$

$$x \doteq 101 \text{ cm}$$