MATHEMATICS 31

A. COURSE OVERVIEW

RATIONALE

To set goals and make informed choices, students need an array of thinking and problem-solving skills. Fundamental to this is an understanding of mathematical techniques and processes that will enable them to apply the basic skills necessary to address everyday mathematical situations, as well as acquire higher order skills in logical analyses and methods for making valid inferences.

A knowledge of mathematics is essential to a well-educated citizenry. However, the need for and use of mathematics in the life of the average citizen are changing. **Emphasis has shifted from the memorization of mathematical formulas and algorithms toward a dynamic view of mathematics as a precise language of terms and symbolic notation, used to describe, reason, interpret and explore.** There is still a need for the logical development of concepts and skills as a basis for the appropriate use of mathematical information to solve problems. Problem-solving strategies, combined with techniques such as estimation and simulation, and incorporated with modern technology, are the tools with which mathematical problems are solved.

Students need procedural competence with numbers and operations, together with a facility in basic algebraic operations and an understanding of fundamental mathematical concepts. They also need to understand the connections among related concepts and be familiar with their use in relevant applications. Students must be able to solve problems, using the mathematical processes developed, and be confident in their ability to apply known mathematical skills and concepts in the acquisition of new mathematical knowledge. In addition, the ability of technology to provide quick and accurate computation and manipulation, to enhance conceptual understanding and to facilitate higher order thinking, should be recognized and used by students.

Mathematics programs must anticipate the changing needs of society, and provide students with the mathematical concepts, skills and attitudes necessary to cope with the challenges of the future.

The majority of students who are enrolled in senior high school mathematics have not yet completely internalized formal thinking with regard to mathematics. They are expected to acquire much abstract understanding in senior high school mathematics courses. The content of each course in the mathematics program is designed to be cognitively appropriate and presented in a way that is consistent with the students' abilities to understand.

The senior high school mathematics program includes the course sequences:

- Mathematics 10–20–30 and Mathematics 31
- Mathematics 13–23–33
- Mathematics 14–24
- Mathematics 16–26.

Mathematics 31 is generally taken after Mathematics 30; however, Mathematics 31 and Mathematics 30 may be taken concurrently.

Mathematics 10–20–30 and Mathematics 31 are designed for students who have achieved the acceptable standard in Mathematics 9, and who are intending to pursue studies beyond high school at a university or in a mathematics-intensive program at a technical school or college. Mathematics 10–20–30 emphasizes the theoretical development of topics in algebra, geometry, trigonometry and statistics up to a level acceptable for entry into such programs.

Having successfully achieved the acceptable standard in each of Mathematics 10, Mathematics 20 and Mathematics 30, students will have met or exceeded the basic mathematics entry requirements for most university or university transfer programs, and will have met or exceeded all mathematics entry requirements for technical school or college programs. A significant number of students would benefit by taking a course in calculus, thus improving their opportunity for success in higher level courses in mathematics at university, such as those required by mathematics and physics honours programs, or by engineering and business programs.

Mathematics 31 emphasizes the theoretical and practical development of topics in the algebra of functions, trigonometry, differential calculus and integral calculus up to a standard acceptable for entry into all first-year programs in mathematics, science, engineering and business. The course is designed to bridge the gap between the Mathematics 10–20–30 course sequence and the calculus course sequences offered by post-secondary institutions.

GOALS

As a culmination of the Mathematics 10–20–30 and Mathematics 31 course sequence, the **specific purposes** of the Mathematics 31 course of studies are as follows:

- to develop an understanding of the algebra of functions and transformations, together with their graphs, and to apply these understandings in different areas of mathematics
- to develop a fluency in algebraic computations involving rational expressions, inequalities, absolute values and trigonometric functions
- to introduce the principal concepts and methods of differential and integral calculus
- to develop skills in problem solving and reasoning, using calculus concepts and procedures as the context
- to apply the methods of calculus to various simple applications in the physical and biological sciences, in engineering, in business and in the social sciences.

PHILOSOPHY

Following the principles elaborated in the *Curriculum and Evaluation Standards for School Mathematics*, National Council of Teachers of Mathematics, 1989, the major focuses of the Mathematics 31 course of studies are on **attitudes**, **problem solving**, **communication**, **reasoning**, **connections** and **technology**. In this document, the term **mathematical processes** is used to make reference to these focuses. Further elaborations of these focuses are given below.

Attitudes

Students are expected to develop an appreciation for the beauty and power of mathematics as a method of solving problems and as an expression of human thought. From this appreciation, they should respect evidence and data, show open-mindedness toward the points of view of others, endeavour to be creative in thinking, persevere in the face of difficulties, be curious in their view of the world, and show a desire to understand. They should show respect for and appreciation of the roles of science and technology in the learning and doing of mathematics.

Problem Solving

Students develop a deep understanding of mathematical concepts and procedures when they solve problems related to practical contexts, to contexts within mathematics itself, and to other school subjects. Students are expected to decide on which concepts and procedures are likely to be useful in solving a particular problem, plan the appropriate application of concepts and procedures to the situation, carry out the plan, and evaluate the reasonableness of the solution and solution procedure when the task is complete. They should then be expected to modify the solution procedure to solve a closely related problem, or apply the solution procedure to a new situation.

Communication

As well as being expected to construct solutions to problems, students are expected to communicate both solutions and solution procedures. They are expected to communicate the representations used for concepts, any approximations or limitations that are inherent in the solution, the reliability of the final solution, and the reasons why particular methods were used. These communications may be oral or written. Students should be able to use the language and symbolism of mathematics appropriately.

Reasoning

A major part of mathematics is reasoning. Students are expected to analyze situations, make assumptions, make and test conjectures, draw conclusions and justify solution procedures. The level of reasoning may vary from speculative to formally rigorous, and students are expected to be able to reason at all of these levels, depending on the context and the purpose set in the problem.

Connections

Students are expected to make connections among mathematics concepts, between mathematics and other school subjects, and between mathematics and everyday life. They are required especially to link new learning to the mathematics they already know, and to apply both old and new mathematics to a wide variety of problem situations. They are expected to make connections between algebraic and geometric representations, and between numerical and symbolic methods. In addition, they are expected to be able to communicate and justify the connections made in a problem solution.

Technology

Students are expected to be aware of the capabilities of various technologies, such as scientific calculators, including those with graphing capabilities; graphing computer software and productivity software, such as word processing, spreadsheet, database and symbol manipulation programs. From this awareness, students are expected to use technology appropriate to the task at hand. Besides the use of technology for precise solutions, students are expected to provide estimates, approximations and sketches, using pencil and paper methods, with no technological aids.

EXPECTATIONS

Students are expected to be mathematically literate at the conclusion of their senior high school mathematics education. **Mathematical literacy** refers to students' abilities and inclinations to manage the demands of their world through the use of mathematical concepts and procedures to solve problems, communicate and reason. More specifically, *students will be expected to:*

- have developed the concepts, skills and attitudes that will enable the acquisition of mathematical knowledge beyond the conclusion of secondary education
- have developed the concepts, skills and attitudes that will ensure success in mathematical situations that occur in future educational endeavours, employment and everyday life
- have achieved understanding of the basic mathematical concepts, and developed the skills and attitudes needed to become responsible and contributing members of society

- apply basic mathematical concepts and skills in practical situations
- have developed critical and creative thinking skills
- communicate mathematical ideas effectively
- understand how mathematics can be used to investigate, interpret and make decisions in human affairs
- understand how mathematics can be used in the analyses of natural phenomena
- understand the connections and interplay among various mathematical concepts and between mathematics and other disciplines
- understand and appreciate the positive contributions of mathematics, as a science and as an art, to civilization and culture.

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B. LEARNER EXPECTATIONS

COURSE FOCUS

The Mathematics 31 course is designed to introduce students to the mathematical methods of calculus. The course acts as a link between the outcomes of the Mathematics 10–20–30 program and the requirements of the mathematics encountered in post-secondary programs. The course builds on existing skills in working with functions and expands this knowledge to include a study of limits in preparation for a study of differential and integral calculus. The methods of calculus are applied to problems encountered in the areas of science, engineering, business and other fields of endeavour.

The focus of the course is to examine functions that describe changing situations as opposed to the more static situations encountered in previous mathematics courses. Emphasis is placed on using graphical methods to illustrate the way in which changing functions behave.

COURSE STRUCTURE

Mathematics 31 is designed in a required–elective format. The **required component** is intended to take the larger proportion of the instructional time. There are eight units available in the **elective component**, of which one or more units are intended to take the remainder of the instructional time.

The time given to the required component, and the number of elective units covered, will vary, depending on local needs. In general, most students will do one or two elective units; however, some students must do as many as four in order to integrate the requirements of external agencies into the Mathematics 31 course.

Required Component

The four sections of the required component are as follows:

- precalculus and limits
- derivatives and derivative theorems
- applications of derivatives
- integrals, integral theorems and integral applications.

Elective Component

The eight possible units available in the elective component are as follows:

- calculus of exponential and logarithmic functions
- numerical methods
- volumes of revolution
- applications of calculus to physical sciences and engineering
- applications of calculus to biological sciences
- applications of calculus to business and economics
- calculus theorems
- further methods of integration.

COURSE LEARNER EXPECTATIONS

The learner expectations found on pages 8 through 51 are the required student outcomes at the end of the Mathematics 31 course. **They do not serve as a required sequence for instruction.** The sequence for instruction will vary, depending on the mathematical background of the students, the learning resources being used, and the preferences of individual teachers.

The learner expectations are divided into *course general learner expectations* and *related specific learner expectations*.

The *course general learner expectations* for Mathematics 31, on pages 8 and 9, provide a bridge between the philosophy statements for all senior high school mathematics courses and the *related specific learner expectations* for each section or unit of study.

The *related specific learner expectations* refer to particular areas of study, such as limits and antiderivatives. They are on pages 10 through 35 for the **required component**, and on pages 36 through 51 for the **elective component**.

Learner expectations are organized into five concept strands, each with three outcome columns. The strands are identified as Mathematical Processes, Measurement and Geometry, Algebra of Functions and Limits, Calculus of Derivatives and Integrals, and Data Management. The outcomes are organized under

Conceptual Understanding, Procedural Knowledge and Problem-solving Contexts.

Although itemized separately, the outcomes are meant to be developed together, using an appropriate balance for the content being studied, and applying the learnings in an appropriate context.

Strands

The strand, **Mathematical Processes**, describes problem-solving, communication and reasoning strategies that apply to many areas of mathematics. Examples include the construction and use of mathematical models and the proper use of approximation techniques for difficult calculations.

In the context of the Mathematics 31 course, mathematical processes include the making and testing of conjectures, the comparison of numerical and algebraic solutions, the connections between similar solution procedures and completely different problems, and the differences between intuitive and rigorous proofs.

The strand, **Measurement and Geometry**, describes the mathematics of spatial reasoning. Measurement involves associating numerical values to various geometrical properties of one-, two- or three-dimensional figures while the geometry involves the properties of one-, two- or three-dimensional figures, and the geometrical representation of domains, limits, derivatives and integrals.

In the context of the Mathematics 31 course, measurement and geometry involve the generalizations of the concepts of slope, curve, line and average to contexts characterized by change, and to contexts where the traditional descriptions give misleading interpretations.

The strand, **Algebra of Functions and Limits**, describes the generalization of patterns, and its description in ordered pair, symbolic and geometrical forms.

In the context of the Mathematics 31 course, the algebra of functions and limits represents an extension of algebraic concepts to enable algebraic models to be used in contexts where functions may not exist, or may be discontinuous.

The strand, **Calculus of Derivatives and Integrals**, describes the extension of algebra to situations that model change.

In the context of the Mathematics 31 course, the calculus of derivatives and integrals refers only to the differential and integral calculus of relations and functions of a single, real variable. The emphasis is more on the use and application of derivatives, and less on the use and application of integrals.

The strand, **Data Management**, describes the mathematics of information processing, from the description of simple data sets to the making of predictions based on models constructed from the data.

In the context of the Mathematics 31 course, data management is confined to the generalization of the concepts of average value and equation root, together with the development of tools that can compute numerical approximations to any desired degree of accuracy. The calculus of random variables is not part of Mathematics 31, though the calculus methods developed in Mathematics 31 could be used in the context of a random variable and its associated probability distribution.

Outcomes

The **Conceptual Understanding** outcomes refer to the major concepts involved in the development of understanding the algebra of functions, the establishment of limits, the differential calculus and the integral calculus.

Conceptual expectations are characterized by the use of such action verbs as *defining*, *demonstrating*, *describing*, *developing*, *establishing*, *explaining*, *giving*, *identifying*, *illustrating*, *linking*, *recognizing and representing*.

The **Procedural Knowledge** outcomes refer to the development of fluency in carrying out procedures related to the general learner expectations. Examples of such procedures are those used for the computation of limits, derivatives and integrals, the determination of maxima and minima, and the computation of average values of functions over intervals.

Procedural expectations are characterized by the use of such action verbs as *approximating*, *calculating*, *constructing*, *differentiating*, *estimating*, *factoring*, *locating*, *rationalizing*, *simplifying*, *sketching*, *solving*, *using* and *verifying*.

The **Problem-solving Contexts** outcomes refer to the construction of models describing applications in a broad range of contexts. They also refer to the combination of algorithms to determine more complicated combinations of limits, derivatives and integrals, and to the use of appropriate technology to approximate difficult models or to carry out long, computational procedures. They also refer to the investigation of phenomena, the collection and interpretation of data in various contexts, and the transfer of knowledge from mathematics to other areas of human endeavour.

Problem-solving expectations are characterized by the use of such action verbs as *adapting*, *analyzing*, *combining*, *communicating*, *comparing*, *connecting*, *constructing*, *deriving*, *evaluating*, *fitting*, *investigating*, *justifying*, *modelling*, *proving*, *reconstructing* and *translating*.

COURSE GENERAL LEARNER EXPECTATIONS BY STRAND

STRANDS	CONCEPTUAL UNDERSTANDING
	Students will demonstrate understanding of the concepts of Mathematics 31, by:
Mathematical Processes Mastery of *course* expectations enables students to construct and solve models describing mathematical situations in a broad range of contexts, and to use the appropriate technology to approximate difficult models and carry out long calculations.	• identifying and giving examples of the strengths and limitations of the use of technology in the computation of limits, derivatives and integrals • giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals.
Measurement and Geometry Mastery of *course* expectations enables students to demonstrate the conceptual underpinnings of calculus to determine limits, derivatives, integrals, rates of change and averages, using geometrical representations.	• describing the relationships among functions after performing translations, reflections, stretches and compositions on a variety of functions • linking displacement, velocity and acceleration of an object moving in a straight line with nonuniform velocity.
Algebra of Functions and Limits Mastery of *course* expectations enables students to translate among symbolic, diagram and graphical representations of situations that describe both continuous and discrete functions of one real variable.	• giving examples of the limits of functions and sequences, both at finite and infinite values of the independent variable.
Calculus of Derivatives and Integrals Mastery of *course* expectations enables students to determine limits, derivatives, integrals and rates of change.	• connecting the derivative with a particular limit, and expressing this limit in situations like secant and tangent lines to a curve • relating the derivative of a complicated function to the derivative of simpler functions • relating the zeros of the derivative function to the critical points on the original curve • recognizing that integration can be thought of as an inverse operation to that of finding derivatives.
Data Management Mastery of *course* expectations enables students to use calculus concepts to describe data distributions and random variables at an introductory level.	• describing the connections between the operation of integration and the finding of areas and averages.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures of Mathematics 31, by:	*Students will* demonstrate problem-solving skills in Mathematics 31, by:
• combining and modifying familiar solution procedures to form a new solution procedure to a related problem.	• using the connections between a given problem and either a simpler or equivalent problem, or a previously solved problem, to solve the given problem • producing approximate answers to complex calculations by simplifying the models used.
• drawing the graphs of functions by applying transformations to the graphs of known functions • calculating the displacement, velocity and acceleration of an object moving in a straight line with nonuniform velocity.	• constructing geometrical proofs at an intuitive level that generalize the concepts of slope, area, average and rate of change.
• expressing final algebraic and trigonometric answers in a variety of equivalent forms, with the form chosen to be the most suitable form for the task at hand • computing limits of functions, using definitions, limit theorems and calculator/computer methods.	• constructing mathematical models for situations in a broad range of contexts, using algebraic and trigonometric functions of a single real variable.
• computing derivatives of functions, using definitions, derivative theorems and calculator/ computer methods • computing definite and indefinite integrals of simple functions, using antiderivatives, integral theorems and calculator/computer methods.	• determining the optimum values of a variable in various contexts, using the concepts of maximum and minimum values of a function • using the concept of critical values to sketch the graphs of functions, and comparing these sketches to computer-generated plots of the same functions • connecting a derivative and the appropriate rate of change, and using this connection to relate complicated rates of change to simpler ones.
• calculating the mean value of a function over an interval.	• fitting mathematical models to situations described by data sets.

RELATED SPECIFIC LEARNER EXPECTATIONS FOR REQUIRED COMPONENT

PRECALCULUS AND LIMITS (Required)

Overview

The first parts of this Mathematics 31 section are designed to reinforce and extend algebraic and trigonometric skills. The algebra of functions introduced in Mathematics 20 and applied to exponential, logarithmic, trigonometric and polynomial functions in Mathematics 30 is extended to the general function. Algebraic and geometric representations of transformations are used. The skills acquired are a necessary adjunct to all work in the Mathematics 31 course. The last part concentrates on the numerical and geometrical development of the concepts of limits and limit theorems.

Student Outcomes Summary

After completing this section, students will be expected to have acquired reliability and fluency in the algebraic skills of factoring, operations with radicals and radical expressions, coordinate geometry, and transformation of functions. They will also be expected to solve linear, quadratic and absolute value inequalities, and to solve trigonometric equations and identities. They will be expected to express the concept of a limit numerically and geometrically, and to calculate limits, using first principles and the limit theorems.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that functions, as well as variables, can be combined, using operations, such as addition and multiplication, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of the algebra of functions, by:
• describing the relationship among functions after performing translations, reflections, stretches and compositions on a variety of functions • drawing the graphs of functions by applying transformations to the graphs of known functions • expressing final algebraic and trigonometric answers in a variety of equivalent forms, with the form chosen to be the most suitable form for the task at hand • constructing mathematical models for situations in a broad range of contexts, using algebraic and trigonometric functions of a single real variable.	• illustrating different notations that describe functions and intervals • expressing, in interval notation, the domain and range of functions • expressing the sum, product, difference and quotient, algebraically and graphically, given any two functions • expressing, algebraically and graphically, the composition of two or more functions --- • illustrating the solution sets for linear, quadratic and absolute value inequalities $\lvert P(x)\rvert \geq a$ $\lvert P(x)\rvert \leq a$ $ax^2 + bx + c \geq d$ --- • illustrating the difference between the concepts of equation and identity in trigonometric contexts.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with the algebra of functions, by:	*Students will* demonstrate problem-solving skills, by:
• using open, closed and semi-open interval notation • finding the sum, difference, product, quotient and composition of functions	• modelling problem situations, using sums, differences, products and quotients of functions • investigating the connections between the algebraic form of a function *f(x)* and the symmetries of its graph.
• solving inequalities of the types $\lvert P(x)\rvert \geq a$ $\frac{\lvert P(x)\rvert}{\lvert Q(x)\rvert} \geq a$ $\lvert P(x)\rvert \leq a$ $\frac{P(x)}{Q(x)} \geq a$ $ax^2 + bx + c \geq d$	
• using the following trigonometric identities: • primary and reciprocal ratio • double and half angle • sum and difference sin $(A \pm B)$ cos $(A \pm B)$ • Pythagorean to simplify expressions and solve equations, express sums and differences as products, and rewrite expressions in a variety of equivalent forms.	

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that functions can be transformed, and these transformations can be represented algebraically and geometrically, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of the transformation of functions, by:
• describing the relationship among functions after performing translations, reflections, stretches and compositions on a variety of functions • drawing the graphs of functions by applying transformations to the graphs of known functions • expressing final algebraic and trigonometric answers in a variety of equivalent forms, with the form chosen to be the most suitable form for the task at hand • constructing mathematical models for situations in a broad range of contexts, using algebraic and trigonometric functions of a single real variable.	• describing the similarities and differences between the graphs of $y = f(x)$ and $y = af[k(x+c)]+d$, where a, k, c and d are real numbers • describing the effects of the reflection of the graphs of algebraic and trigonometric functions across any of the lines $y = x$, $y = 0$, or $x = 0$ • describing the effects of the parameters a, b, c and d on the trigonometric function $f(x) = a \sin [b(x+c)]+d$
	• describing the relationship between parallel and perpendicular lines • describing the condition for tangent, normal and secant lines to a curve • linking two problem conditions to a system of two equations for two unknowns.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with the transformation of functions, by:	*Students will* demonstrate problem-solving skills, by:
• sketching the graph of, and describing algebraically, the effects of any translation, reflection or dilatation on any of the following functions or their inverses: • linear, quadratic or cubic polynomial • absolute value • reciprocal • exponential • step • sketching and describing, algebraically, the effects of any combination of translation, reflection or dilatation on the following functions: • $f(x) = a \sin [b(x+c)]+d$ • $f(x) = a \cos [b(x+c)]+d$ • $f(x) = a \tan [b(x+c)]$	• translating problem conditions into equation or inequality form.
• finding the equation of a line, given any two conditions that serve to define it • solving systems of linear–linear, linear–quadratic or quadratic–quadratic equations.	

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that final answers may be expressed in different equivalent forms, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of equivalent forms, by:
• giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals • expressing final algebraic and trigonometric answers in a variety of equivalent forms, with the form chosen to be the most suitable form for the task at hand.	• describing what it means for two algebraic or trigonometric expressions to be equivalent.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with the construction of equivalent forms, by: • factoring expressions with integral and rational exponents, using a variety of techniques • rationalizing expressions containing a numerator or a denominator that contains a radical • simplifying rational expressions, using any of the four basic operations.	*Students will* demonstrate problem-solving skills, by: • illustrating the difference between verification and proof in the comparison of two algebraic or trigonometric expressions.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that descriptions of change require a careful definition of limit and the precise use of limit theorems, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of limits and limit theorems, by:
• identifying and giving examples of the strengths and limitations of the use of technology in the computation of limits, derivatives and integrals • giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals • giving examples of the limits of functions and sequences, both at finite and infinite values of the independent variable • computing limits of functions, using definitions, limit theorems and calculator/ computer methods.	• explaining the concept of a limit • giving examples of functions with limits, with left-hand or right-hand limits, or with no limit • giving examples of bounded and unbounded functions, and of bounded functions with no limit • explaining, and giving examples of, continuous and discontinuous functions --- • defining the limit of an infinite sequence and an infinite series --- • explaining the limit theorems for sum, difference, multiple, product, quotient and power • illustrating, using suitable examples, the limit theorems for sum, difference, multiple, product, quotient and power.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with limits and limit theorems, by:	*Students will* demonstrate problem-solving skills, by:
• determining the limit of any algebraic function as the independent variable approaches finite or infinite values for continuous and discontinuous functions • sketching continuous and discontinuous functions, using limits, intercepts and symmetry	• proving and illustrating geometrically, the following trigonometric limits: $\lim_{x \to 0} \frac{\sin x}{x}$ or $\lim_{x \to 0} \frac{x}{\sin x} = 1$ AND $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
• calculating the sum of an infinite convergent geometric series	• using the basic trigonometric limits, combined with limit theorems, to determine the limits of more complex trigonometric expressions
• using definitions and limit theorems to determine the limit of any algebraic function as the independent variable approaches a fixed value • using definitions and limit theorems to determine the limit of any algebraic function as the independent variable approaches $\pm\infty$.	• comparing numerical and algebraic processes for the determination of algebraic and trigonometric limits

DERIVATIVES AND DERIVATIVE THEOREMS (Required)

Overview

The concept of the slope of a line can be extended to give students an intuitive feel for the meaning of slope at a point on a curve. By examining slopes of secants, and using the ideas of limits, the derivative can be put into a well-defined form. Derivative theorems are used to find only the derivatives of algebraic and trigonometric functions.

Student Outcomes Summary

After completing this section, students will be expected to find derivatives of algebraic and trigonometric functions and determine equations of tangents at specific points. For selected functions, they will also be expected to prove these derivatives, using first principles. Students use the power, chain, product and quotient rules to calculate the derivatives of these functions. They will also be expected to use implicit differentiation to find the derivatives of certain algebraic and trigonometric relations.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that the derivative of a function is a limit that can be found, using first principles, and demonstrate this, by: • giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals • connecting the derivative with a particular limit, and expressing this limit in situations like secant and tangent lines to a curve • computing derivatives of functions, using definitions, derivative theorems and calculator/computer methods.	*Students will* demonstrate conceptual understanding of derivatives, by: • showing that the slope of a tangent line is a limit • explaining how the derivative of a polynomial function can be approximated, using a sequence of secant lines • explaining how the derivative is connected to the slope of the tangent line - • recognizing that $f(x) = x^n$ can be differentiated where $n \in R$ • identifying the notations $f'(x)$, y', and $\frac{dy}{dx}$ as alternative notations for the first derivative of a function • explaining the derivative theorems for sum and difference $(f \pm g)'(x) = f'(x) \pm g'(x)$ • explaining the sense of the derivative theorems for sum and difference, using practical examples.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with derivatives, by:	*Students will* demonstrate problem-solving skills, by:
• finding the slopes and equations of tangent lines at given points on a curve, using the definition of the derivative • estimating the numerical value of the derivative of a polynomial function at a point, using a sequence of secant lines	• deriving $f'(x)$ for polynomial functions up to the third degree, using the definition of the derivative • using the definition of the derivative to find $f'(x)$ for $f(x) = (ax + c)^n$ where $n = -1$ or $n = \frac{1}{2}$.
• using the definition of the derivative to determine the derivative of $f(x) = x^n$ where n is a positive integer • differentiating polynomial functions, using the derivative theorems for sum and difference • differentiating functions that are single terms of the form x^n where n is rational.	

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that derivatives of more complicated functions can be found from the derivatives of simpler functions, using derivative theorems, and demonstrate this, by: • giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals • relating the derivative of a complicated function to the derivative of simpler functions • expressing final algebraic and trigonometric answers in a variety of equivalent forms, with the form chosen to be the most suitable form for the task at hand • computing derivatives of functions, using definitions, derivative theorems and calculator/ computer methods.	*Students will* demonstrate conceptual understanding of derivative theorems, by: • demonstrating that the chain, power, product and quotient rules are aids to differentiate complicated functions • identifying implicit differentiation as a tool to differentiate functions where one variable is difficult to isolate • explaining the relationship between implicit differentiation and the chain rule - • comparing the sum, difference, product and quotient theorems for limits and derivatives • explaining the derivation of the derivative theorems for product and quotient $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$ • explaining the derivative theorems for product and quotient, using practical examples - • illustrating second, third and higher derivatives of algebraic functions • describing the second derivative geometrically.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with derivative theorems, by: • finding the derivative of a polynomial, power, product or quotient function • applying the chain rule in combination with the product and quotient rule • using the technique of implicit differentiation - - - - - - - - • writing final answers in factored form • finding the slope and equations of tangent lines at given points on a curve - - - - - - - - • finding the second and third derivatives of functions.	*Students will* demonstrate problem-solving skills, by: • deriving the quotient rule from the product rule • showing that equivalent forms of the derivative of a rational function can be found by using the product and the quotient rules • determining the derivative of a function expressed as a product of more than two factors • determining the second derivative of an implicitly defined function AND, in one or more of the following, by: • showing the derivative for a relation found by both implicit and explicit differentiation to be the same • finding the equations of tangent lines to the standard conics.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that trigonometric functions have derivatives, and these derivatives obey the same derivative theorems as algebraic functions, and demonstrate this, by: • identifying and giving examples of the strengths and limitations of the use of technology in the computation of limits, derivatives and integrals • giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals • relating the derivative of a complicated function to the derivative of simpler functions • computing derivatives of functions, using definitions, derivative theorems and calculator/computer methods.	*Students will* demonstrate conceptual understanding of the derivatives of trigonometric functions, by: • demonstrating that the three primary trigonometric functions have derivatives at all points where the functions are defined • explaining how the derivative of a trigonometric function can be approximated, using a sequence of secant lines.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with derivatives of trigonometric functions, by: • calculating the derivatives of the three primary and three reciprocal trigonometric functions • estimating the numerical value of the derivative of a trigonometric function at a point, using a sequence of secant lines • using the power, chain, product and quotient rules to find the derivatives of more complicated trigonometric functions • using the derivative of a trigonometric function to calculate its slope at a point, and the equation of the tangent at that point.	*Students will* demonstrate problem-solving skills, by: • using the definition of the derivative to find the derivative for the sine and cosine functions • explaining why radian measure has to be used in the calculus of trigonometric functions.

APPLICATIONS OF DERIVATIVES
(Required)

Overview

Inside the area of pure mathematics, graph sketching of polynomial functions, rational algebraic functions and simple trigonometric functions are looked at. Derivatives are used to solve maximum and minimum problems in economic and geometrical contexts and to solve related rate problems in geometrical and algebraic contexts.

Student Outcomes Summary

After completing this section, students will be expected to provide systematic sketches of graphs of polynomial, rational, algebraic and trigonometric functions of one variable. They will be expected to solve maximum–minimum and related rate problems in economic and geometric contexts. They will be expected to construct maximum–minimum and related rate models in geometric contexts, and to use the techniques of derivatives to solve problems based on the models constructed.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that calculus is a powerful tool in determining maximum and minimum points and in sketching of curves, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of maxima and minima, by:
• relating the zeros of the derivative function to the critical points on the original curve • using the connections between a given problem and either a simpler or equivalent problem, or a previously solved problem, to solve the given problem • constructing mathematical models for situations in a broad range of contexts, using algebraic and trigonometric functions of a single real variable • determining the optimum values of a variable in various contexts, using the concepts of maximum and minimum values of a function • using the concept of critical values to sketch the graphs of functions, and comparing these sketches to computer-generated plots of the same functions • fitting mathematical models to situations described by data sets.	• identifying, from a graph sketch, locations at which the first and second derivative are zero • illustrating under what conditions symmetry about the *x*-axis, *y*-axis or the origin will occur • explaining how the sign of the first derivative indicates whether or not a curve is rising or falling; and by explaining how the sign of the second derivative indicates the concavity of the graph - - - - - - - - - - • illustrating, by examples, that a first derivative of zero is one possible condition for a maximum or a minimum to occur • explaining circumstances wherein maximum and minimum values occur when $f'(x)$ is not zero • illustrating, by examples, that a second derivative of zero is one possible condition for an inflection point to occur • explaining the differences between local maxima and minima and absolute maxima and minima in an interval - - - - - - - - - - • explaining when finding a maximum value is appropriate and when finding a minimum value is appropriate.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with maxima and minima, by:	*Students will* demonstrate problem-solving skills, by:
• sketching the graphs of the first and second derivative of a function, given its algebraic form or its graph • using zeros and intercepts to aid in graph sketching	• employing a systematic calculus procedure to sketch algebraic and trigonometric functions • comparing and contrasting graphs plotted on a calculator and graphs sketched, using a systematic calculus procedure • calculating maxima and minima for such quantities as volumes, areas, perimeters and costs • illustrating the connections among geometric, economic or motion problems, the modelling equations of these problems, the resulting critical points on the graphs and their solutions, using derivatives
• using the first and second derivatives to find maxima, minima and inflection points to aid in graph sketching • determining vertical, horizontal and oblique asymptotes, and domains and ranges of a function • finding intervals where the derivative is greater than zero or less than zero in order to predict where the function is increasing or decreasing • verifying whether or not a critical point is a maximum or a minimum	AND, in one or more of the following, by: • constructing a mathematical model to represent a geometric problem, and using the model to find maxima and/or minima • constructing a mathematical model to represent a problem in economics, and using the model to find maximum profits or minimum cost
• using a given model, in equation or graph form, to find maxima or minima that solve a problem.	• constructing a mathematical model to represent a motion problem, and using the model to find maximum or minimum time or distance.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that complicated rates of change can be related to simpler rates, using the chain rule, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of related rates, by:
• relating the derivative of a complicated function to the derivative of simpler functions • combining and modifying familiar solution procedures to form a new solution procedure to a related problem • using the connections between a given problem and either a simpler or equivalent problem, or a previously solved problem, to solve the given problem • constructing geometrical proofs at an intuitive level that generalize the concepts of slope, area, average and rate of change • expressing the connection between a derivative and the appropriate rate of change, and using this connection to relate complicated rates of change to simpler ones.	• illustrating how the chain rule can be used to represent the relationship between two or more rates of change • explaining the clarity that Leibnitz's notation gives to expressing related rates --- • illustrating the time rate of change of a function $y = f(x)$ or a relation $R(x, y) = 0$.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with related rates, by:	*Students will* demonstrate problem-solving skills, by:
• using the chain rule to find the derivative of a function with respect to an external variable, such as time • using Leibnitz's notation to illustrate related rates	• constructing a mathematical model to represent time rates of change of linear measures, areas, volumes, surface areas, etc. • solving related rate problems that use models containing primary trigonometric functions.
• constructing a chain rule of related rates, using appropriate variables • calculating related rates for the time derivatives of areas, volumes, surface areas and relative motion • calculating related rates of change with respect to time, given an equation that models electronic circuits or other engineering situations • using the chain rule to derive an acceleration function in terms of position, given a velocity function expressed in terms of position.	

INTEGRALS, INTEGRAL THEOREMS AND INTEGRAL APPLICATIONS (Required)

Overview

Corresponding to the operation of finding derivatives is the inverse operation of finding the antiderivatives. The area under a curve can be defined as the limit of the sum of small rectangular areas and can be found from this definition. The fundamental theorem of calculus links the solution of the area problem to the finding of special antiderivatives. As in the case of derivatives, the use of integral theorems makes the finding of antiderivatives and areas much easier. Derivatives and integrals can be interrelated through the context of nonuniform motion in a straight line.

Student Outcomes Summary

After completing this section, students will be expected to find the antiderivatives of any polynomial function, together with the antiderivatives of the simplest rational, algebraic and trigonometric functions. They will be expected to use antiderivatives to find areas under curves, and to use the connection between derivatives and integrals to relate displacement, velocity and acceleration. Conceptually, students will be expected to illustrate the properties of antiderivatives, definite integrals and the fundamental theorem of calculus, using geometric representations.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that the operation of finding a derivative has an inverse operation of finding an antiderivative, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of antiderivatives, by:
• giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals • recognizing that integration can be thought of as an inverse operation to that of finding derivatives • computing definite and indefinite integrals of simple functions, using antiderivatives, integral theorems and calculator/computer methods • using the connections between a given problem and either a simpler or equivalent problem, or a previously solved problem, to solve the given problem.	• explaining how differentiation can have an inverse operation • showing that many different functions can have the same derivative • representing, on the same grid, a family of curves that form a sequence of functions, all having the same derivative.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with antiderivatives, by:	*Students will* demonstrate problem-solving skills, by:
• finding the antiderivatives of polynomials, rational algebraic functions and trigonometric functions	• finding the antiderivatives of rational functions and polynomial powers by comparison and inspection methods • determining antiderivatives for polynomial, rational and trigonometric functions.
• finding the family of curves whose first derivative has been given • solving separable first order differential equations for general and specific solutions.	

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that the area under a curve can be expressed as the limit of a sum of smaller rectangles, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of area limits, by:
• identifying and giving examples of the strengths and limitations of the use of technology in the computation of limits, derivatives and integrals • giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals • constructing geometrical proofs at an intuitive level that generalize the concepts of slope, area, average and rate of change.	• defining the area under a curve as a limit of the sums of the areas of rectangles • establishing the existence of upper and lower bounds for the area under a curve.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with area limits, by:	*Students will* demonstrate problem-solving skills, by:
• sketching the area under a curve (polynomials, rational, trigonometric) over a given interval, and approximating the area as the sum of individual rectangles.	• communicating the similarities and differences between definite integrals and areas under curves.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that the area under a curve can be related to the antiderivative of the function defining the curve; and in turn, can use the antiderivative to determine the areas under and between curves, and demonstrate this, by: • giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals • recognizing that integration can be thought of as an inverse operation to that of finding derivatives • describing the connections between the operation of integration with the finding of areas and averages • computing definite and indefinite integrals of simple functions, using antiderivatives, integral theorems and calculator/computer methods • calculating the mean value of a function over an interval • constructing geometrical proofs at an intuitive level that generalize the concepts of slope, area, average and rate of change.	*Students will* demonstrate conceptual understanding of definite integrals, by: • identifying the indefinite integral $\int f(x)dx$ as a sum of an antiderivative $F(x)$ and a constant c • explaining how the definite integral between fixed limits a and b is a number whose value is $F(b) - F(a)$ - • explaining the connection between the numerical values of the area and the definite integral for functions f of a constant sign and a variable sign • illustrating the following properties of integrals $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ - • describing the sense of the fundamental theorem of calculus • explaining how the fundamental theorem of calculus relates the area limit to the antiderivative of a function describing a curve.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS

Students will demonstrate competence in the procedures associated with definite integrals, by:
 	
• calculating the definite integral for polynomial, rational and trigonometric functions
 	
- - - - -
 	
• determining the area between a curve and the x-axis over a given domain
 	
• determining the area between a curve and the x-axis:
  • if $f(x)$ has a constant sign over a given interval
  • if $f(x)$ has a change in sign over a given interval
 	
- - - - -
 	
• determining the area between curves over a given interval
 	
• determining the area between intersecting curves.
	Students will demonstrate problem-solving skills, by:
	
	• relating the value of the integral between $x = a$ and $x = b$ to the area between the curve and the x-axis over the interval $[a, b]$
	
	• using integration theorems, such as those listed below, to simplify definite integrals
	$\int_a^b f(x)dx = -\int_b^a f(x)dx$
	$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
	
	• using integral theorems to simplify more complicated integrals
	
	• illustrating the conditions necessary for a function to be differentiable, or able to be integrated.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that velocity and acceleration are the first and second derivatives of displacement with respect to time; and that once one of the three functions is known, the other two can be found by finding derivatives or antiderivatives, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of the relationships among displacement, velocity and acceleration, by:
• linking displacement, velocity and acceleration of an object moving in a straight line with nonuniform velocity	• describing the motion of a body, using sketches of the first and second derivatives
• calculating the displacement, velocity and acceleration of an object moving in a straight line with nonuniform velocity • calculating the mean value of a function over an interval • constructing geometrical proofs at an intuitive level that generalize the concepts of slope, area, average and rate of change.	• explaining the difference between a stationary point and a turning point in the context of linear motion • illustrating the concepts of derivative and antiderivative in the context of displacement, velocity and acceleration.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with displacement, velocity and acceleration, by:	*Students will* demonstrate problem-solving skills, by:
• estimating an instantaneous velocity, using slopes of secant lines to represent average velocities	• solving problems associated with distance, velocity and acceleration whose models are restricted to those of the forms $y'(t) = f(t)$ and $y''(t) = f(t)$
• finding the first and second derivatives of a position function to get instantaneous velocity and instantaneous acceleration functions, where the position function is an algebraic function or a trigonometric function of time • using antiderivatives of acceleration and velocity functions to get velocity and displacement functions.	• deriving the following kinematic equations, starting from the expression a = constant: • $v = at + v_0$ • $v^2 = v_0^2 + 2ad$ • $d = ½at^2 + v_0 t + d_0$ • determining the equations of velocity and acceleration in simple harmonic motion, starting from the displacement equation: • $x = A \cos (kt + c)$.

RELATED SPECIFIC LEARNER EXPECTATIONS FOR ELECTIVE COMPONENT

CALCULUS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS (Elective)

Overview

The calculus of exponential and logarithmic functions has many important applications in modelling natural phenomena, such as natural growth and decay, and problems involving return to equilibrium, such as those found in thermal and chemical reactions. The study of natural logarithms and the natural base exponential function is key in the development of derivatives and integrals for all such functions, as well as in their applications. This elective, together with the elective Applications of Calculus to Biological Sciences, can be taught as an integrated whole.

Student Outcomes Summary

After completing this elective section, students will be expected to represent symbolically and graphically, differentiate and integrate, exponential and logarithmic functions. They will also be expected to solve problems where the modelling equation is given, and to construct equations to model a situation with given parameters.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that exponential and logarithmic functions have limits, derivatives and integrals that obey the same theorems as do algebraic and trigonometric functions, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of the calculus of exponential and logarithmic functions, by:
• computing limits of functions, using definitions, limit theorems and calculator/ computer methods • computing derivatives of functions, using definitions, derivative theorems and calculator/computer methods • computing definite and indefinite integrals of simple functions, using antiderivatives, integral theorems and calculator/computer methods • constructing mathematical models for situations in a broad range of contexts, using algebraic and trigonometric functions of a single real variable • fitting mathematical models to situations described by data sets.	• defining exponential and logarithmic functions as inverse functions • explaining the special properties of the number *e*, together with a definition of *e* as a limit --- • illustrating that the derivative of an exponential or logarithmic function may be derived from the definition of the derivative --- • illustrating that base-*e* exponential and logarithmic functions form a convenient framework within which the calculus of similar functions in any base may be developed --- • illustrating how exponential and logarithmic functions may be used to model certain natural problems involving growth, decay and return to equilibrium.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with the calculus of exponential and logarithmic functions, by:	*Students will* demonstrate problem-solving skills, by:
• estimating the values of the limits *e* and e^x	• evaluating maxima and minima of given functions involving exponential and logarithmic functions • finding areas bounded by exponential, logarithmic or reciprocal functions
• approximating the slopes of $y = e^x$ and $y = \ln x$ for some specific value of *x* • finding the derivative and antiderivative of the base-*e* exponential function • using limit theorems to evaluate the limits of simple exponential and logarithmic functions • finding the derivative of the natural logarithmic function	AND, in one or more of the following, by: • relating natural growth, natural decay and return to equilibrium to the differential equations $y' = ky$ or $y' = k(y - y_0)$ • solving natural growth and decay problems, starting from the equations $y' = ky$ or $y' = k(y - y_0)$
• finding the derivatives of logarithmic functions having bases other than *e* • finding the derivatives and antiderivatives of exponential functions having bases other than *e*	• fitting exponential models to observed data • calculating distance, velocity and acceleration for falling bodies, with air resistance present as part of the model
• computing $\int_a^b \frac{1}{x}\,dx$ and $\int_a^b e^{kx}\,dx$ • solving the differential equations $y' = ky$ *and* $y' = k(y - y_0)$.	• connecting the integral $\int_a^b \frac{dx}{px + q}$ with the integral $\int_a^b \frac{dx}{x}$.

NUMERICAL METHODS (Elective)

Overview

Limits and derivatives are sometimes difficult to compute using algebraic methods, but often can be calculated more easily using numerical approximations and estimates. In the case of definite integrals, most functions cannot be integrated exactly, and numerical methods are necessary. This unit has a balance between intuitive methods based on guess-and-test, and formal procedures, such as Simpson's rule for definite integrals.

Student Outcomes Summary

After completing this elective section, students will be expected to compute numerical approximations for limits, equation roots and definite integrals. They will be expected to relate these procedures to the fundamental concepts of calculus studied in the required sections of Mathematics 31, and to apply these concepts to a variety of relations, functions and equations. Finally, they will be expected to evaluate the reliability of a numerical procedure and to communicate the results of the evaluation.

<table>
<tr><th>GENERAL LEARNER EXPECTATIONS</th><th>CONCEPTUAL UNDERSTANDING</th></tr>
<tr>
<td rowspan="3">Students are expected to understand that many limits, derivatives, equation roots and definite integrals can be found numerically, and demonstrate this, by:

• identifying and giving examples of the strengths and limitations of the use of technology in the computation of limits, derivatives and integrals

• computing limits of functions, using definitions, limit theorems and calculator/ computer methods

• computing derivatives of functions, using definitions, derivative theorems and calculator/computer methods

• computing definite and indefinite integrals of simple functions, using antiderivatives, integral theorems and calculator/computer methods.</td>
<td>Students will demonstrate conceptual understanding of the principles of numerical analysis, by:

• describing the difference between an exact solution and an approximate solution

• identifying when a particular numerical method is likely to give poor results</td>
</tr>
<tr><td>• explaining the difference between iterative and noniterative procedures

• explaining the basis of the Newton–Raphson procedure for determining the roots of $f(x) = 0$</td></tr>
<tr><td>• describing the basis of a limit, derivative, equation root or integral procedure in geometric terms

• connecting the number of subdivisions of the range of integration with the accuracy of the estimate for the integral

• showing that all numerical integration formulas are procedures that interpolate between the lower and upper Riemann sums for the integral.</td></tr>
</table>

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with numerical methods, by:	*Students will* demonstrate problem-solving skills, in one or more of the following:
• estimating the value of a limit by systematic trial and error • calculating the numerical value of the derivative at a point on a curve, whether or not there is a defining formula for the curve	• comparing the errors in computing definite integrals when using different procedures • writing computer software for the computation of limits, equation roots or definite integrals • reconstructing limit processes so that numerical evaluations can be efficient and reliable • evaluating the reliability of a numerical procedure for finding a limit, an equation root or a definite integral.
• solving the equation $f(x) = 0$ by systematic trial and error, and by the Newton–Raphson method	
• calculating the upper and lower Riemann sums for a definite integral • calculating the value of a definite integral, using the midpoint rule • calculating the value of a definite integral, using the trapezoidal rule • calculating the value of a definite integral, using Simpson's rule.	

VOLUMES OF REVOLUTION (Elective)

Overview

Volumes of solids may be calculated using triple integrals, but for volumes formed by revolving the graphs of simple functions about the *x*- or *y*-axis, a single integration will suffice. By using limits to find an infinite sum of cylindrical volumes, a value for a volume of revolution can be found.

Student Outcomes Summary

After completing this elective section, students will be expected to find volumes of revolutions of the graphs of simple polynomial and trigonometric functions by using the "disc" method. Students will be expected to relate this procedure to that of finding the area between the graph of a function and the *x*-axis by the limit of the sum of a set of rectangles whose widths approach zero (a Riemann sum).

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that volumes of revolution may be considered as the limiting sum of smaller volumes and can be related to definite integrals, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of volumes of revolution, by:
• describing the connections between the operation of integration and the finding of areas and averages • combining and modifying familiar solution procedures to form a new solution procedure to a related problem • computing definite and indefinite integrals of simple functions, using antiderivatives, integral theorems and calculator/computer methods • using the connections between a given problem and either a simpler or equivalent problem, or a previously solved problem, to solve the given problem.	• identifying the solid generated by the rotation of the graph of a function, either between two boundary values, or between two graphs • explaining the connection between the volume of revolution and the volume of a cylindrical disc • demonstrating how the formula for the volume of revolution by the disc method could be generated.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with volumes of revolution, by: • using the relationship $V = \pi\int_a^b [f(x)]^2\,dx$ to find the volume of revolution between the boundaries of *a* and *b* for polynomial and trigonometric functions • finding the volume of revolution between two polynomial or trigonometric functions by first finding the intersection points of the graphs of the two functions.	*Students will* demonstrate problem-solving skills, in one or both of the following, by: • deriving formulas for the volume of a cylinder, cone and sphere • revolving the graph of a function about a horizontal or vertical line, other than the *x*- or *y*-axis, and finding the resulting volume of revolution.

APPLICATIONS OF CALCULUS TO PHYSICAL SCIENCES AND ENGINEERING
(Elective)

Overview

Calculus is used extensively in physical sciences and in engineering. The study of energy, forces and motion can be represented in terms of differential equations that involve algebraic and trigonometric functions of time. These differential equation models, together with their solutions, can be used to make hypotheses and predictions.

Student Outcomes Summary

After completing this elective section, students will be expected to construct differential equation models for simple physical science and engineering situations, and to solve those models that involve algebraic and trigonometric functions of time.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that most of the important equations of physics are differential equations, and calculus provides the most efficient solution method, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of the links among calculus, the physical sciences and engineering, by:
• linking displacement, velocity and acceleration of an object moving in a straight line with nonuniform velocity • describing the connections between the operation of integration with the finding of areas and averages • calculating the displacement, velocity and acceleration of an object moving in a straight line with nonuniform velocity • calculating the mean value of a function over an interval • producing approximate answers to complex calculations by simplifying the models used • constructing mathematical models for situations in a broad range of contexts, using algebraic and trigonometric functions of a single real variable • fitting mathematical models to situations described by data sets.	• illustrating situations in which differential equations are required to represent problems • developing one or more differential equations in the areas of linear motion, simple harmonic motion, work, hydrostatic force, moments of inertia, radioactive decay or similar situations • showing that the concept of mean value can be applied to situations where the quantity varies with time, or where a range of values exists in a system of multiple bodies.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with the application of calculus to the physical sciences and engineering, by:	*Students will* demonstrate problem-solving skills, in one of the following, by:
• solving differential equations of type $y''(t) = f(t)$	• determining the half-life, the decay rate, and the activity as a function of time for a radioisotope
• solving differential equations of type $y''(t) = -k^2 y$	• analyzing the motion and energy in an oscillating spring system (Hooke's law) • analyzing hydrostatic forces on the surface of submerged objects
• calculating the work done by any nonuniform force *f(x)* using $W = \int_a^b f(x)dx$ • determining root–mean–square values for sinusoidal functions.	• determining the moment of inertia for rigid bodies • determining the centre of mass for individual bodies and for systems of bodies • integrating Newton's second law when expressed in the form $mx''(t) = f(x)$.

APPLICATIONS OF CALCULUS TO BIOLOGICAL SCIENCES (Elective)

Overview

Many biological problems are concerned with rates of growth, rates of decay, and rates of transport of energy and materials across boundaries. Mathematical solutions to many of these problems can be found by constructing differential equation models that have solutions involving exponential and logarithmic functions. This elective, together with the elective Calculus of Exponential and Logarithmic Functions, can be taught as an integrated whole.

Student Outcomes Summary

After completing this elective section, students will be expected to construct differential equation models for simple biological situations, and to solve those models that involve exponential and logarithmic functions.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that many important biological applications of calculus are connected with models involving the solution of the differential equation $f'(x) = kf(x)$, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of the links between calculus and the biological sciences, by:
• combining and modifying familiar solution procedures to form a new solution procedure to a related problem • using the connections between a given problem and either a simpler or equivalent problem, or a previously solved problem, to solve the given problem • producing approximate answers to complex calculations by simplifying the models used • constructing mathematical models for situations in a broad range of contexts, using algebraic and trigonometric functions of a single real variable • fitting mathematical models to situations described by data sets.	• defining exponential and logarithmic functions as inverse functions • explaining the special properties of the number *e*, together with a definition of *e* as a limit • illustrating that the derivative of an exponential or logarithmic function may be derived from the definition of the derivative - • illustrating how differential equations may be used to model certain biological problems involving growth, decay and movement across a boundary.

<table>
<tr><th>PROCEDURAL KNOWLEDGE</th><th>PROBLEM-SOLVING CONTEXTS</th></tr>
<tr><td>

Students will demonstrate competence in the procedures associated with the application of calculus to the biological sciences, by:

- estimating the values of the limits e and e^x
- using limit theorems to evaluate the limits of simple exponential and logarithmic functions
- approximating the slopes of $y = e^x$ and $y = \ln x$ for some specific value of x
- finding the derivative of the natural logarithmic function

- finding the derivative and antiderivative of the base-e exponential function
- using methods of guess-and-test, and comparing coefficients to solve the differential equations $y' = ky$ and $y' = k(y - y_0)$
- verifying that $y = Ae^{kx}$ satisfies the differential equation $y' = ky$.

</td><td>

Students will demonstrate problem-solving skills, by:

- relating natural growth, natural decay and return to equilibrium to the differential equations $y' = ky$ or $y' = k(y - y_0)$
- solving natural growth and decay problems starting from the equations $y' = ky$ or $y' = k(y - y_0)$

AND, either, by:

- fitting differential equation models to observed biological data

OR, both of the following, by:

- relating growth subject to limits to the logistic equation $y' = ky(L - y)$
- solving logistic equation models, and estimating values for the parameters k and L, from experimental data.

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</table>

APPLICATIONS OF CALCULUS TO BUSINESS AND ECONOMICS (Elective)

Overview

Although there are far too many variables and influences present to make a complete analysis of economic situations, many of them can be simplified to a point where Mathematics 31 students can begin to understand the applications of calculus to this area of study.

Student Outcomes Summary

After completing this elective section, students will be expected to analyze an economic model based on the need to maximize revenue and profit while minimizing cost.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that calculus may be used as a tool to analyze situations in business and economics that involve revenue, profit and cost, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of the links among calculus, business and economics, by:
• relating the zeros of the derivative function to the critical points on the original curve • producing approximate answers to complex calculations by simplifying the models used • constructing mathematical models for situations in a broad range of contexts, using algebraic and trigonometric functions of a single real variable • determining the optimum values of a variable in various contexts, using the concepts of maximum and minimum values of a function • using the concept of critical values to sketch the graphs of functions, and comparing these sketches to computer-generated plots of the same functions • fitting mathematical models to situations described by data sets.	• explaining how calculus procedures frequently arise in models used in business and economics • explaining how both maximum and minimum values are important in the making of business and economic decisions.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with the application of calculus to business and economics, by:	*Students will* demonstrate problem-solving skills, in one or both of the following, by:
• sketching polynomial, exponential and trigonometric functions of one variable	• determining a revenue, profit or cost function from a problem situation that can be modelled, using polynomial functions
• using a given revenue, profit or cost function to calculate and justify optimum values • finding the maximum of a revenue or profit function that is expressed as a function of price or number sold • finding the minimum of a cost function that is expressed as a function of price or number sold.	• modelling the business cycle, using trigonometric functions.

CALCULUS THEOREMS (Elective)

Overview

The simple geometric representations of limit, continuity, derivative and integral work well for most functions used in problem applications. However, it is possible to give examples of misleading conclusions when these representations are being used. This elective section shows these examples and leads the student into a deeper knowledge, and the proofs, of the basic limit, derivative and integral theorems.

Student Outcomes Summary

After completing this elective section, students will be expected to illustrate the differences between intuitive proofs and rigorous proofs in the context of proof, construct a counterexample to a hypothesis, and construct a rigorous proof of one of the simpler limit, derivative or integral theorems.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that limit, derivative and integral theorems can be proved at different levels of rigor, from intuitive to analytic, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of the nature of proof in the context of limit, derivative and integral theorems, by:
• giving examples of the differences between intuitive and rigorous proofs in the context of limits, derivatives and integrals • calculating the mean value of a function over an interval • constructing geometrical proofs at an intuitive level that generalize the concepts of slope, area, average and rate of change.	• proving the equivalence of the product and the quotient rule for derivatives • comparing the nature of intuitive and rigorous proofs knowing the conditions under which a theorem is true • explaining what is an example, and what is a counterexample • illustrating the intermediate value theorem, Rolle's theorem, the mean value theorem and the fundamental theorem of calculus, by examples and counterexamples, using both graphical and algebraic formulations.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with the construction of proofs of limit, derivative and integral theorems, by: • using specific functions to illustrate the equivalence of the product and quotient rules • locating an error in a given proof of a calculus theorem • verifying the mean value theorem and the fundamental theorem of calculus for specific examples.	*Students will* demonstrate problem-solving skills, by: • composing rigorous proofs for derivative theorems, starting from the corresponding limit theorems AND, in one of the following, by: • deriving formulas for the derivatives of complicated functions, using the basic rules; e.g., functions, such as $y = \frac{f(x)g(x)}{h(x)}$ or $y = f(g(h(x)))$ • proving that a differentiable function satisfies the mean value theorem • constructing, and justifying, an iterative solution procedure for the equation $f(x) = c$, using the intermediate value theorem.

FURTHER METHODS OF INTEGRATION
(Elective)

Overview

Certain algebraic and trigonometric functions cannot be integrated using the usual methods involving the power rule. It is therefore necessary to look at other methods; these being integration by substitution, integration by parts, or integration by partial fractions.

Student Outcomes Summary

After completing this elective section, students will be expected to recognize which particular method of integration is needed. They will also be expected to use and combine these methods appropriately in evaluating with indefinite and definite integrals.

GENERAL LEARNER EXPECTATIONS	CONCEPTUAL UNDERSTANDING
Students are expected to understand that integration by substitution, by parts, and by partial fractions are procedures necessary when dealing with certain algebraic and trigonometric functions, and demonstrate this, by:	*Students will* demonstrate conceptual understanding of the methods of integration, by:
• recognizing that integration can be thought of as an inverse operation to that of finding derivatives • expressing final algebraic and trigonometric answers in a variety of equivalent forms, with the form chosen to be the most suitable form for the task at hand • computing definite and indefinite integrals of simple functions, using antiderivatives, integral theorems and calculator/computer methods • using the connections between a given problem and either a simpler or equivalent problem, or a previously solved problem, to solve the given problem.	• identifying integrals that cannot be evaluated, using the antiderivatives of polynomial or trigonometric functions • showing how substitutions into a definite integral change both the function to be integrated and the limits of integration • recognizing when it would be appropriate to use integration by substitution, by parts or by partial fractions.

PROCEDURAL KNOWLEDGE	PROBLEM-SOLVING CONTEXTS
Students will demonstrate competence in the procedures associated with methods of integration, by:	*Students will* demonstrate problem-solving skills, by:
	• deriving the formula for integration by parts, using the formula for the derivative of a product
• using a change of variable to integrate by substitution • using a trigonometric substitution to integrate algebraic functions containing terms, such as $\sqrt{a^2 - x^2}$ • using a trigonometric identity as a first step in an integration by substitution	• combining two or more methods to integrate a rational algebraic function • adapting integration methods for antiderivatives to the evaluation of definite integrals.
• using integration by parts to integrate a product • writing a rational function as a sum of partial fractions • using integration by partial fractions to integrate rational algebraic functions.	